Adaptive finite difference solution for fractional boundary value problems

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Outline

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Homogeneous Dirichlet BVP of the fractional Laplacian

Main task: We are concerned with the numerical solution of homogeneous Dirichlet boundary value problem (BVP):

$$\begin{cases} (-\Delta)^s u = f, & \text{in } \Omega \\ u = 0, & \text{in } \Omega^c \equiv \mathbb{R}^d \setminus \Omega \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 1$) and $(-\Delta)^s$ is the fractional Laplacian (operator) of order $s \in (0, 1)$.

Representations of the fractional Laplacian:

- Fourier transform representation

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\boldsymbol{\xi}|^{2s}\mathcal{F}(u))$$

- Singular integral representation

$$\begin{aligned}(-\Delta)^s u(\boldsymbol{x}) &= C_{d,s} \text{ p.v. } \int_{\mathbb{R}^d} \frac{u(\boldsymbol{x}) - u(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{d+2s}} d\boldsymbol{y} \\ &= C_{d,s} \text{ p.v. } \int_{\Omega} \frac{u(\boldsymbol{x}) - u(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{d+2s}} d\boldsymbol{y} + C_{d,s} u(\boldsymbol{x}) \int_{\Omega^c} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^{d+2s}} d\boldsymbol{y}\end{aligned}$$

- Several other representations ...

Homogeneous Dirichlet BVP of the fractional Laplacian

Regularity of BVP's solution:

- Ros-Oton and Serra (2014):

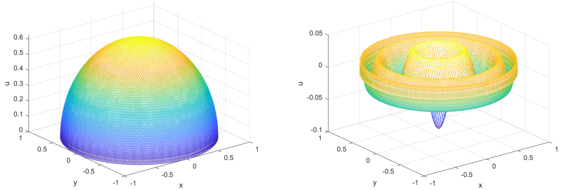
$$u(\boldsymbol{x}) \sim dist(\boldsymbol{x}, \partial\Omega)^s, \quad \text{near } \partial\Omega$$
$$u(\boldsymbol{x}) \in C^\infty, \quad \text{in interior}$$

- Ros-Oton and Serra (2014):

$$\|u\|_{C^s(\mathbb{R}^d)} \le C\|f\|_{L^\infty(\Omega)}$$

- Acosta and Borthagaray (2017):

$$|u|_{H^{s+\frac{1}{2}-0}(\Omega)} \le \begin{cases} C\|f\|_{C^{\frac{1}{2}-s}(\Omega)}, & 0 < s < \frac{1}{2} \\ C\|f\|_{L^\infty(\Omega)}, & s = \frac{1}{2} \\ C\|f\|_{C^\beta(\Omega)}, & \frac{1}{2} < s < 1 \quad \text{for some } \beta > 0 \end{cases}$$



(a) $s = 0.5$ $(k = 0)$

(b) $s = 0.5$ $(k = 5)$

Why consider fractional Laplacian, fractional-order differential equations?

Motivations:

- Interesting in theory: 1st-order, 2nd-order, ... why not 0.5th-order or 1.5th-order?
- Statistical mechanics: the concentration of particles performing Brownian motion (random walks with short-range jumps) follows the standard diffusion equation while the concentration of particles performing Lévy flights (random walks with long-range jumps) satisfies a fractional diffusion equation.
- Quantum mechanics: Klein-Gordon operators $\sqrt{-\Delta + m^2}$ (tempered fractional Laplacian)
- It has been reported that fractional models can give more accurate description of underlying phenomena in image processing, finance, and biology, especially for anomalous dynamics (compared to dynamics satisfying Gaussian distribution).
- Good tool for use to model global interaction, long-range decay, and multiple scales.

References (Books, theses, and Reviews, incomplete list):

- I. Podlubny: Fractional Differential Equations, Academic Press, Inc., San Diego, CA, 1999.
- K. Diethelm: The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin, 2010.
- M. K. Ishteva: Properties and Applications of the Caputo Fractional Operator. Master Thesis, Universität Karlsruke, 2005.
- Lischke et al., JCP (2020): "What is the fractional Laplacian? A comparative review with new results"

Outline

Numerical solution of homogeneous Dirichlet problems

- Main challenges in numerical solution:
  - High cost: the stiffness matrix is a full matrix
  - Slow convergence (less than or equal to 1st order), due to low regularity of the solution
- It has attracted considerable attention in the last decade from researchers and a variety of methods have been developed: Finite element, finite difference, spectral, meshfree, discrete Galerkin, Monte Carlo methods).

Finite difference methods (incomplete list)

- H. Wang and T. Basu (SIAM J. Sci. Comput., 2012): 1D, based on Grünwald-Letnikov formula
- Y. Huang and A. Oberman (SIAM J. Numer. Anal., 2014): 1D FDMs, prove $\mathcal{O}(h^{2-2s})$ in L_∞ for smooth solutions and $\mathcal{O}(h^s)$ for non-smooth solutions
- S. Duo, H. van Wyk, and Y. Zhang (J. Comput. Phys., 2018): new 1D scheme and prove $\mathcal{O}(h^2)$ for smooth solutions
- N. Du, H. Sun, and H. Wang (Comput. Appl. Math., 2019): a volume penalized finite difference scheme and a preconditioned Krylov subspace iterative algorithm
- Z. Hao and R. Du (J. Comput. Phys., 2021): FDM and $\mathcal{O}(h^2)$ for smooth solutions in L_∞.
- R. Han and S. Wu (SIAM J. Numer. Anal., 2022): prove $\mathcal{O}(|\log h|h^{2-s})$ for $0 < s < 1$ and $\mathcal{O}(h^{2s})$ for $s < 2/3$ in L_∞
- M. Chen, W. Deng, C. Min, J. Shi, and M. Stynes (2023): prove $N^{-\min(rs, 2-2s)}$ on graded meshes (DOI: 10.13140/RG.2.2.10784.15361)

Finite element methods (incomplete list)

- G. Acosta and J. Borthagaray (SIAM J. Numer. Anal. 2017): linear finite elements, prove $\mathcal{O}(h^{\frac{1}{2}-0})$ for uniform meshes and $\mathcal{O}(N^{-\frac{1}{d}})$ (for $1/2 < s < 1$) for graded meshes in H^s
- G. Acosta, F. Bersetche, and J. Borthagaray (Comput. Math. Appl. 2017): 2D FEM code
- G. Acosta, J. Borthagaray, and N. Heuer (IMA J. Numer. Anal. 2018): nonhomogeneous Dirichlet problems
- J. Borthagaray, L. Del Pezzo, and S. Martínez (J. Sci. Comput. 2018): $\mathcal{O}(h^{\min(1,s+1/2)-0})$ for quasi-uniform meshes and $\mathcal{O}(N^{-(1+s)/d})$ for graded meshes in L^2. Eigenvalue problems.
- M. Ainsworth and C. Glusa (Comput. Methods Appl. Mech. Engrg. 2017, Contemporary computational mathematics 2018): a sparse approximation to the stiffness matrix and an efficient multigrid implementation. $\mathcal{O}(N^{-(1+s)/d})$ for adaptive meshes in L^2
- M. Faustmann, M. Karkulik, and J. Melenk (SIAM J. Numer. Anal. 2022): Local convergence for FEM

Numerical solution of homogeneous Dirichlet problems

Spectral, DG, collocation, and meshfree methods (incomplete list)

- G. Pang, W. Chen, and Z. Fu (J. Comput. Phys. 2015): RBF collocation (meshfree)
- X. Zhang, M. Gunzburger, L. Ju (Comput. Methods Appl. Mech. Engrg. 2016): Collocation method
- F. Song, C. Xu, and G. Karniadakis (SIAM J. Sci. Comput. 2017): spectral method for spectral fractional Laplacian
- Q. Du, L. Ju, and J. Lu (Math. Comp. 2019): DG for time dependent problems
- H. Antil, P. Dondl, and L. Striet (SIAM J. Sci. Comput. 2021): sinc function method for spectral fractional Laplacian
- J. Burkardt, Y. Wu, and Y. Zhang (SIAM J. Sci. Comput. 2021): meshfree pseudospectral method
- H. Li, R. Liu and L. Wang (Numer. Math. Theory Methods Appl. 2022): Hermite spectral-Galerkin method

Outline

Finite element approximation

Weak formulation

$$\frac{C_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(\boldsymbol{x})-u(\boldsymbol{y}))(v(\boldsymbol{x})-v(\boldsymbol{y}))}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2s}}d\boldsymbol{x}d\boldsymbol{y}=\int_\Omega fvd\boldsymbol{x},\quad \forall v\in \tilde{H}^s(\Omega)$$

or

$$\frac{C_{d,s}}{2}\int_{\Omega}\int_{\Omega}\frac{(u(\boldsymbol{x})-u(\boldsymbol{y}))(v(\boldsymbol{x})-v(\boldsymbol{y}))}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2s}}d\boldsymbol{x}d\boldsymbol{y}$$
$$+C_{d,s}\int_{\Omega}\int_{\Omega^c}\frac{u(\boldsymbol{x})v(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2s}}d\boldsymbol{x}d\boldsymbol{y}=\int_\Omega fvd\boldsymbol{x},\quad \forall v\in \tilde{H}^s(\Omega)$$

Features of finite element approximation:

- Need to compute $2d$-dimensional singular integrals and integrals on unbounded domains.
- Full stiffness matrix: costly to assemble
- Full stiffness matrix: costly to carry out its multiplication with vectors: $\mathcal{O}(N_v^2)$ flops.
  - 2D: for a mesh of size $10^2\times 10^2$, $N_v=10^4$ and $N_v^2=10^8$
  - 3D: for a mesh of size $10^2\times 10^2\times 10^2$, $N_v=10^6$ and $N_v^2=10^{12}$
- Works for arbitrary bounded domains and with mesh adaptation
- Sharp error estimates available: $\mathcal{O}(h^{\min(1,s+\frac{1}{2})-0})$ for quasi-uniform meshes and $\mathcal{O}(\bar{h}^{1+s})$ for graded and adaptive meshes in L^2, $\bar{h}=N^{-1/d}$

Outline

Uniform-grid FD approximation based on Fourier transform

Fourier transform representation:

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\boldsymbol{\xi}|^{2s}\mathcal{F}(u))$$

In 2D:

$$(-\Delta)^s u(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{(-\Delta)^s} u(\xi,\eta) e^{ix\xi} e^{iy\eta} d\xi d\eta$$

$$\widehat{(-\Delta)^s} u(\xi,\eta) = (\xi^2+\eta^2)^s \hat{u}(\xi,\eta)$$

Discrete Fourier transform (DFT): Consider a uniform infinite grid (lattice)

$$(x_j, y_k) = (jh_{\text{FD}}, kh_{\text{FD}}),\ j,k \in \mathbb{Z},$$

DFT and its inverse on this grid are given by

$$\breve{u}(\xi,\eta) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} u_{j,k} e^{-ix_j\xi} e^{-iy_k\eta}$$

$$u(x_j,y_k) = \frac{h_{\text{FD}}^2}{(2\pi)^2}\int_{-\frac{\pi}{h_{\text{FD}}}}^{\frac{\pi}{h_{\text{FD}}}}\int_{-\frac{\pi}{h_{\text{FD}}}}^{\frac{\pi}{h_{\text{FD}}}} \breve{u}(\xi,\eta) e^{ix_j\xi} e^{iy_k\eta} d\xi d\eta$$

$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \breve{u}(\frac{\xi}{h_{\text{FD}}},\frac{\eta}{h_{\text{FD}}}) e^{ij\xi} e^{ik\eta} d\xi d\eta$$

Uniform-grid FD approximation based on Fourier transform

Consider the 5-point FD approximation to the Laplacian:

$$(-\Delta_h)u(x_j, y_k) = \frac{1}{h_{\text{FD}}^2}(u_{j+1,k} - 2u_{j,k} + u_{j-1,k}) + \frac{1}{h_{\text{FD}}^2}(u_{j,k+1} - 2u_{j,k} + u_{j,k-1})$$

Applying the DFT to the above equation, we get

$$(-\check{\Delta}_h)u(\xi, \eta) = \frac{1}{h_{\text{FD}}^2}\left(4\sin^2(\frac{\xi h_{\text{FD}}}{2}) + 4\sin^2(\frac{\eta h_{\text{FD}}}{2})\right)\check{u}(\xi, \eta).$$

The FD approximation of the fractional Laplacian is given by

$$(-\Delta_h)^{s}u(x_j, y_k) = \frac{h_{\text{FD}}^2}{(2\pi)^2}\int_{-\frac{\pi}{h_{\text{FD}}}}^{\frac{\pi}{h_{\text{FD}}}}\int_{-\frac{\pi}{h_{\text{FD}}}}^{\frac{\pi}{h_{\text{FD}}}}\frac{1}{h_{\text{FD}}^{2s}}\left(4\sin^2(\frac{\xi h_{\text{FD}}}{2}) + 4\sin^2(\frac{\eta h_{\text{FD}}}{2})\right)^{s}\check{u}(\xi, \eta)e^{ij\xi}e^{ik\eta}d\xi d\eta$$

where

$$\check{u}(\xi, \eta) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}u_{j,k}e^{-ix_j\xi}e^{-iy_k\eta}$$

Uniform-grid FD approximation based on Fourier transform

The FD approximation of the fractional Laplacian:

$$(-\Delta_h)^s u(x_j, y_k) = \frac{1}{h_{\text{FD}}^{2s}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} A_{(j,k),(m,n)} u_{m,n} = \frac{1}{h_{\text{FD}}^{2s}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} T_{j-m,k-n} u_{m,n}$$

$$= \frac{1}{h_{\text{FD}}^{2s}} \sum_{m=-N}^{N} \sum_{n=-N}^{N} T_{j-m,k-n} u_{m,n} \qquad (\text{ for } u = 0 \text{ on } \mathbb{R}^d \setminus \Omega)$$

- $T_{p,q}$'s are the Fourier coefficients of $\left(4\sin^2(\frac{\xi}{2}) + 4\sin^2(\frac{\eta}{2})\right)^s$, i.e.,

$$T_{p,q} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(4\sin^2(\frac{\xi}{2}) + 4\sin^2(\frac{\eta}{2})\right)^s e^{ip\xi} e^{iq\eta} d\xi d\eta$$

$$T_{-p,-q} = T_{p,q}, \quad T_{-p,q} = T_{p,q}, \quad T_{p,-q} = T_{p,q}$$

  They can be approximated using FFT (fast Fourier transform)
- A_{FD} is block Teoplitz matrix, symmetric and positive definite, and

$$\lambda_{\min}(A_{\text{FD}}) \geq C h_{\text{FD}}^{2s}, \quad \lambda_{\max}(A_{\text{FD}}) \leq C$$

- $A_{\text{FD}} \boldsymbol{u}$ can be computed using FFT in $\mathcal{O}(N^d \log N^d)$ flops (almost linear about N^d)
- Works only for rectangular/cubic domains; cannot be incorporated with mesh adaptation.

Outline

GoFD (J. Shen and WH: 2024) for arbitrary bounded domains

Existing methods:

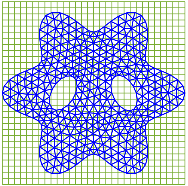
- Finite difference methods: Efficient (using FFT), uniform rectangular meshes, simple domains
- Finite element methods: Slow (with full stiffness matrix), unstructured meshes, arbitrary domains

Aim: develop a method: efficient (via FFT), unstructured meshes, arbitrary domains

$$(-\Delta)^s u = f \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^c$$

$$\Longrightarrow \quad \frac{1}{h_{\text{FD}}^{2s}} D_h^{-1} (I_h^{\text{FD}})^T A_{\text{FD}} I_h^{\text{FD}} \boldsymbol{u} = \boldsymbol{f} \qquad (1)$$

Mesh $\mathcal{T}_h$ (on which BVP is solved) overlaid by Uniform grid $\mathcal{T}_{\text{FD}}$ (on which $(-\Delta)^s$ is discretized)



where

- A_{FD}: uniform-grid FD approx. of $(-\Delta)^s$ on $\mathcal{T}_{\text{FD}}$
- I_h^{FD}: transfer matrix from $\mathcal{T}_h$ to $\mathcal{T}_{\text{FD}}$ (sparse)
- D_h: a diagonal matrix formed by the column sums of I_h^{FD}
- $\boldsymbol{u} = \{u(\boldsymbol{x}_j), j = 1, ..., N_v\}$, $\boldsymbol{f} = \{f(\boldsymbol{x}_j), j = 1, ..., N_v\}$, $\boldsymbol{x}_j$'s: vertices of $\mathcal{T}_h$

(1) can be written into a symmetric system as

$$(I_h^{\text{FD}})^T A_{\text{FD}} I_h^{\text{FD}} \boldsymbol{u} = h_{\text{FD}}^{2s} D_h \boldsymbol{f} \qquad (2)$$

Theorem 1 (J. Shen and WH, 2024)

If I_h^{FD} has full column rank, then D_h is invertible, $(I_h^{FD})^T A_{FD} I_h^{FD}$ is symmetric and positive definite, and thus (2) is solvable.

GoFD (J. Shen and WH: 2024) for arbitrary bounded domains

Theorem 2 (J. Shen and WH, 2024)

Let I_h^{FD} be the transfer matrix associated with piecewise linear interpolation. If the uniform grid's spacing h_{FD} satisfies

$$h_{FD} \le \frac{a_{min}}{(d+1)\sqrt{d}},$$

where a_{min} is the minimum height of $\mathcal{T}_h$, then I_h^{FD} has full column rank. In this case, $(I_h^{FD})^T A_{FD} I_h^{FD}$ is symmetric and positive definite and thus invertible.

Features of GoFD:

- $(I_h^{\text{FD}})^T A_{\text{FD}} I_h^{\text{FD}}$ is symmetric and positive definite and satisfies

$$\lambda_{\min}((I_h^{\text{FD}})^T A_{\text{FD}} I_h^{\text{FD}}) \ge C h_{\text{FD}}^{2s} (a_{\min}/h)^{2d}, \quad \lambda_{\max}((I_h^{\text{FD}})^T A_{\text{FD}} I_h^{\text{FD}}) \le C$$

- $(I_h^{\text{FD}})^T A_{\text{FD}} I_h^{\text{FD}} \boldsymbol{u}$ can be carried out via FFT and therefore,

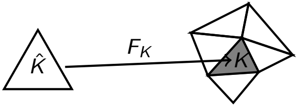
$$(I_h^{\text{FD}})^T A_{\text{FD}} I_h^{\text{FD}} \boldsymbol{u} = h_{\text{FD}}^{2s} D_h \boldsymbol{f}$$

  can be solved efficiently using Krylov subspace iteration methods (e.g. Conjugate Gradient).
- Preconditioners based on ILU (J. Shen and WH 2024).
- Works for any bounded domain and with mesh adaptation.
- Convergence order will be demonstrated by numerical examples.

Outline

Mesh terminology:

- $\mathcal{T}_h = \{K\}$: a simplicial mesh for $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3, ...$)
- N: the number of elements, N_v: the number of vertices, $\hat{K}$: the master element
- $F_K : \hat{K} \to K$ is the affine mapping and F_K' is the Jacobian matrix of F_K.



- Edge matrix of K: $E_K = [\boldsymbol{x}_1^K - \boldsymbol{x}_0^K, ..., \boldsymbol{x}_d^K - \boldsymbol{x}_0^K]$
- Edge matrix of $\hat{K}$: $\hat{E} = [\boldsymbol{\xi}_1 - \boldsymbol{\xi}_0, ..., \boldsymbol{\xi}_d - \boldsymbol{\xi}_0]$
- Relation between F_K', E_K, and $\hat{E}$:

$$\boldsymbol{x} = F_K(\boldsymbol{\xi}) = F_K'(\boldsymbol{\xi} - \boldsymbol{\xi}_0^K) + \boldsymbol{x}_0^K \quad \Longrightarrow \quad E_K = F_K' \hat{E} \quad \Longrightarrow \quad \boxed{F_K' = E_K \hat{E}^{-1}}$$

- $\mathbb{J}_K = (F_K')^{-1} = \hat{E} E_K^{-1}$
- **Use the metric tensor (monitor function) $\mathbb{M} = \mathbb{M}(\boldsymbol{x})$ to control mesh concentration on Ω.**

The MMPDE moving mesh method

Given a meshing energy/functional $I_h[\mathcal{T}_h] = \sum_K |K| G(\mathbb{J}_K, \det(\mathbb{J}_K), \mathbb{M}_K)$, **the MMPDE approach** defines the mesh nodal velocities as a gradient system of the energy:

$$\frac{d\boldsymbol{x}_i}{dt} = -\frac{P_i}{\tau}\left(\frac{\partial I_h}{\partial \boldsymbol{x}_i}\right)^T = \frac{P_i}{\tau}\sum_{K\in\omega_i} |K| \boldsymbol{v}_{i_K}^K, \quad i = 1, ..., N_v$$

- $\tau > 0$ is used to control the response time of the mesh movement to the changes in $\mathbb{M}$
- $P_i > 0$ is chosen to make the equation invariant under the scaling transformation of $\mathbb{M}$
- ω_i is the element patch associated with $\boldsymbol{x}_i$
- i_K is the local index of $\boldsymbol{x}_i$ on K
- $\boldsymbol{v}_{i_K}^K$'s are the local velocities given by

$$\begin{bmatrix} (\boldsymbol{v}_1^K)^T \\ \vdots \\ (\boldsymbol{v}_d^K)^T \end{bmatrix} = -GE_K^{-1} + E_K^{-1}\frac{\partial G}{\partial \mathbb{J}}\hat{E}E_K^{-1} + \frac{\partial G}{\partial \det(\mathbb{J})}\frac{\det(\hat{E})}{\det(E_K)}E_K^{-1} - \frac{1}{d+1}\sum_{j=0}^{d}\text{tr}\left(\frac{\partial G}{\partial \mathbb{M}}\mathbb{M}_{j,K}\right)\begin{bmatrix} (E_K^{-1})_j \\ \vdots \\ (E_K^{-1})_j \end{bmatrix}$$

$$(\boldsymbol{v}_0^K)^T = -\sum_{k=1}^{d}(\boldsymbol{v}_k^K)^T - \sum_{j=0}^{d}\text{tr}\left(\frac{\partial G}{\partial \mathbb{M}}\mathbb{M}_{j,K}\right)(E_K^{-1})_j$$

$$\mathbb{M}_{j,K} = \mathbb{M}(\boldsymbol{x}_j^K), \qquad (E_K^{-1})_j: \text{ the } j\text{-th row of } E_K^{-1}$$

Theorem 6.1 (Kamenski & WH, Math Comput 2018)

Assume that the metric tensor is bounded

$$\underline{m}\,\mathbb{I} \le \mathbb{M}(\boldsymbol{x}) \le \overline{m}\,\mathbb{I}, \quad \forall \boldsymbol{x} \in \Omega$$

for some positive constants $\underline{m}$ and $\overline{m}$, and the meshing functional satisfies the coercivity condition

$$I_h = \sum_K |K| G(\mathbb{J}_K, \det(\mathbb{J}_K), \mathbb{M}_K): \qquad G(\mathbb{J}, \det(\mathbb{J}), \mathbb{M}, \boldsymbol{x}) \ge \alpha \left[\mathrm{tr}(\mathbb{J}\mathbb{M}^{-1}\mathbb{J}^T)\right]^q - \beta, \quad \forall \boldsymbol{x} \in \Omega,$$

with $q > d/2$ and positive constants α and β. If the initial mesh is nonsingular, then

- *The mesh governed by the $\boldsymbol{x}$-formulation of MMPDE will be nonsingular for $t > 0$;*
- *Specifically, the minimal height and volume of K are bounded below by*

$$a_K \ge C(\mathcal{T}_h^0)\, \overline{m}^{-\frac{d}{2(2q-d)} - \frac{1}{2}} N^{-\frac{2q}{d(2q-d)}}, \quad \forall K \in \mathcal{T}_h, \quad \forall t > 0$$

$$|K| \ge C(\mathcal{T}_h^0)\, \overline{m}^{-\frac{d^2}{2(2q-d)} - \frac{d}{2}} N^{-\frac{2q}{(2q-d)}}, \quad \forall K \in \mathcal{T}_h, \quad \forall t > 0$$

1. Holds for fully discrete MMPDE if Δt is sufficiently small (depending on $\overline{m}$ and N).
2. Holds for any (convex or concave) domain in any dimension.
3. Works for the equidistribution–alignment functional and other meshing functionals.

$\mathbb{M}$-uniform mesh approach and equidistribution & alignment:

- $\mathcal{T}_h$ is said to be $\mathbb{M}$-uniform if it is uniform in the metric $\mathbb{M} = \mathbb{M}(\boldsymbol{x})$.
- Equivalently, all of its elements have the same size and are similar to $\hat{K}$, measured in $\mathbb{M}$.
- An $\mathbb{M}$-uniform mesh satisfies the equidistribution condition (for element size)

$$|K|\sqrt{\det(\mathbb{M}_K)} = \frac{1}{N}\sigma_h, \quad \forall K \in \mathcal{T}_h \qquad \left(\sigma_h = \sum_K |K|\sqrt{\det(\mathbb{M}_K)} \right)$$

and the alignment condition (for element similarity – shape & orientation)

$$\frac{1}{d}\,\mathrm{tr}((F_K')^T \mathbb{M}_K F_K') = \det((F_K')^T \mathbb{M}_K F_K')^{\frac{1}{d}}, \quad \forall K \in \mathcal{T}_h$$

- An energy functional associated with these conditions is $(\mathbb{J}_K = (F_K')^{-1}, \quad \mathbb{M}_K = \mathbb{M}(\boldsymbol{x}_K))$

$$I_h[\mathcal{T}_h] = \sum_K |K|\sqrt{\det(\mathbb{M}_K)} \left(\theta \left(\mathrm{tr}(\mathbb{J}_K \mathbb{M}_K^{-1} \mathbb{J}_K^T) \right)^{\frac{dp}{2}} + (1-2\theta) d^{\frac{dp}{2}} \left(\frac{\det(\mathbb{J}_K)}{\sqrt{\det(\mathbb{M}_K)}} \right)^p \right)$$
$$\equiv \sum_K |K| G(\mathbb{J}_K, \det(\mathbb{J}_K), \mathbb{M}_K)$$

where $\theta \in (0, 0.5]$ and $p > 1$ are dimensionless parameters. Magic choice: $\theta = 1/3$, $p = 2$.

The MMPDE moving mesh method

The derivatives of G wrt $\mathbb{J}$, $\det(\mathbb{J})$, and $\mathbb{M}$ can be found using scalar-by-matrix differentiation as

$$
\begin{aligned}
G &= \theta \sqrt{\det(\mathbb{M})} \left(\operatorname{tr}(\mathbb{J}\mathbb{M}^{-1}\mathbb{J}^T) \right)^{\frac{dp}{2}} + (1-2\theta) d^{\frac{dp}{2}} \sqrt{\det(\mathbb{M})} \left(\frac{\det(\mathbb{J})}{\sqrt{\det(\mathbb{M})}} \right)^{p} \\
\frac{\partial G}{\partial \mathbb{J}} &= dp\theta \sqrt{\det(\mathbb{M})} \left(\operatorname{tr}(\mathbb{J}\mathbb{M}^{-1}\mathbb{J}^T) \right)^{\frac{dp}{2}-1} \mathbb{M}^{-1}\mathbb{J}^T \\
\frac{\partial G}{\partial \det(\mathbb{J})} &= p(1-2\theta) d^{\frac{dp}{2}} \det(\mathbb{M})^{\frac{1-p}{2}} \det(\mathbb{J})^{p-1} \\
\frac{\partial G}{\partial \mathbb{M}} &= -\frac{dp\theta}{2} \sqrt{\det(\mathbb{M})} \left(\operatorname{tr}(\mathbb{J}\mathbb{M}^{-1}\mathbb{J}^T) \right)^{\frac{dp}{2}-1} \mathbb{M}^{-1}\mathbb{J}^T\mathbb{J}\mathbb{M}^{-1} \\
&\quad + \frac{\theta}{2} \sqrt{\det(\mathbb{M})} \left(\operatorname{tr}(\mathbb{J}\mathbb{M}^{-1}\mathbb{J}^T) \right)^{\frac{dp}{2}} \mathbb{M}^{-1} \\
&\quad + \frac{(1-p)(1-2\theta) d^{\frac{dp}{2}}}{2} \det(\mathbb{M})^{\frac{1-p}{2}} \det(\mathbb{J})^{p} \mathbb{M}^{-1} \\
P_i &= \det(\mathbb{M}(\boldsymbol{x}_i))^{\frac{p-1}{2}}
\end{aligned}
$$

Outline

Example 1:

$$\begin{cases} (-\Delta)^s u = \frac{2^{2s}\Gamma(1+s+k)\Gamma(\frac{d}{2}+s+k)}{k!\ \Gamma(\frac{d}{2}+k)} \cdot P_k^{s,\frac{d}{2}-1}\big(2|\boldsymbol{x}|^2-1\big), & \text{in } \Omega \equiv B_1(0) \\ u = 0, & \text{in } B_1(0)^c \end{cases}$$

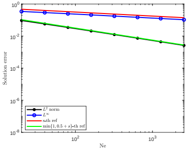
where $B_1(0)$ is the unit ball and $P_k^{s,\frac{d}{2}-1}(\cdot)$ is the Jacobi polynomial of degree k with parameters $(s, \frac{d}{2}-1)$. The exact solution is

$$u = (1-|\boldsymbol{x}|^2)_+^{\alpha} P_k^{\alpha,\frac{d}{2}-1}\big(2|\boldsymbol{x}|^2-1\big)$$

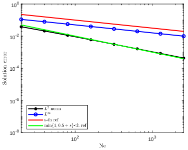
Remarks

- Adaptive meshes are obtained with the Moving Mesh PDE method (MMPDE) (Russell, Ren, and WH 1994 and 2010).
- Results are shown for $k = 0$ and $k = 5$ in 1D, 2D, and 3D

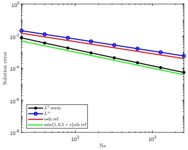
For quasi-uniform meshes: $\mathcal{O}(h^s)$ in L^∞ norm and $\mathcal{O}(h^{\min(1,0.5+s)})$ in L^2 norm, consistent with convergence order proved and numerically demonstrated for finite element approximation



(a) $s = 0.25$

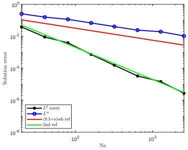


(b) $s = 0.5$

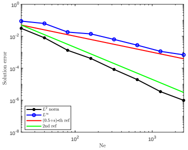


(c) $s = 0.75$

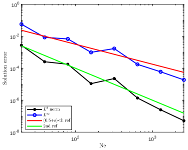
For adaptive meshes: $\mathcal{O}(\bar{h}^{s+0.5})$ in L^∞ norm and $\mathcal{O}(\bar{h}^2)$ in L^2 norm, better than convergence order proved for finite element approximation. $\bar{h} = 1/N$.



(a) $s = 0.25$

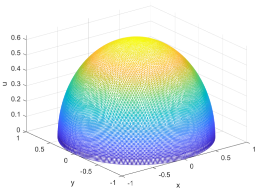


(b) $s = 0.5$

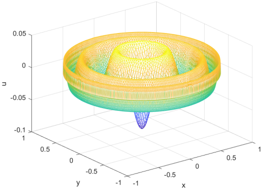


(c) $s = 0.75$

Example 1 - 2D

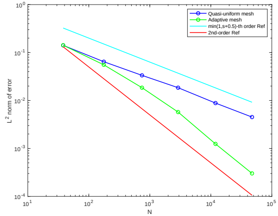


(a) $k = 0$, $s = 0.5$

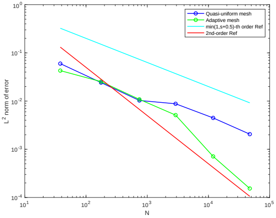


(b) $k = 5$, $s = 0.5$

Error in L^2 norm: $\mathcal{O}(h^{\min(1,0.5+s)})$ for quasi-uniform meshes and $\mathcal{O}(\bar{h}^2)$ for adaptive meshes



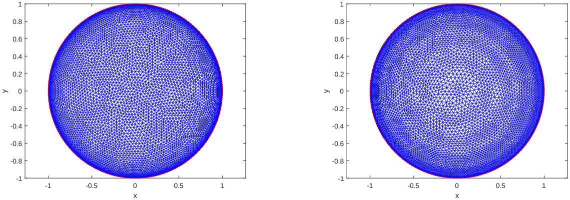
(a) $k = 0$, $s = 0.5$



(b) $k = 5$, $s = 0.5$

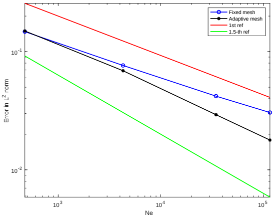
Example 1 - 3D ($s = 0.5$)

Adaptive meshes



(a) $k = 0$ (b) $k = 5$

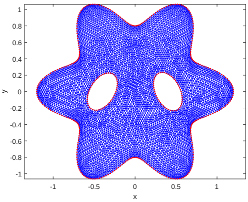
Error in L^2 norm: $\mathcal{O}(h^{\min(1,0.5+s)})$ for quasi-uniform meshes and $\mathcal{O}(\bar{h}^{1+s})$ for adaptive meshes



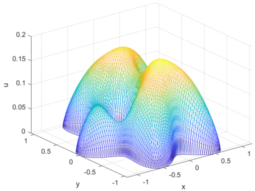
Example 2 - 2D (Batman-shaped domain with $s = 0.75$)

$$\begin{cases} (-\Delta)^s u = 1, & \text{in } \Omega \\ u = 0, & \text{in } \Omega^c \end{cases}$$

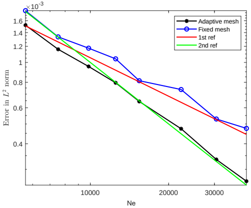
Analytical solution is not available. The reference solution (for computing the error) is obtained with a fine adaptive mesh. Complex domain with holes inside and waving exterior boundary.



(a) Adaptive mesh



(b) Computed solution



(c) Solution error

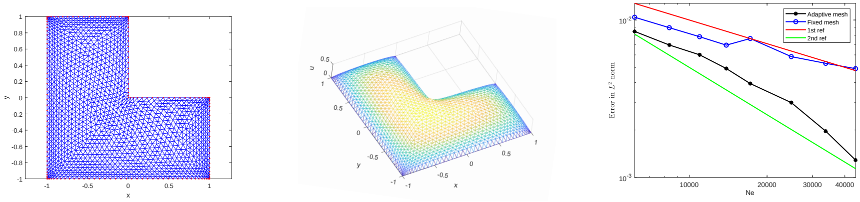
The error in L^2 norm:

- $\mathcal{O}(h^{\min(1,0.5+s)})$ for quasi-uniform meshes
- $\mathcal{O}(\bar{h}^2)$ for adaptive meshes

Example 3 - 2D (L-shaped domain with $s = 0.5$)

$$\begin{cases} (-\Delta)^s u = 1, & \text{in } \Omega \\ u = 0, & \text{in } \Omega^c \end{cases}$$

Analytical solution is not available. The reference solution (for computing the error) is obtained with a fine adaptive mesh.



(a) Adaptive mesh (b) Computed solution (c) Solution error

The error in L^2 norm:

- $\mathcal{O}(h^{\min(1,0.5+s)})$ for quasi-uniform meshes
- $\mathcal{O}(\bar{h}^2)$ for adaptive meshes

Outline

Conclusions

GoFD for Dirichlet problems of the fractional Laplacian:

- Relatively simple to code
- Efficient implementation via FFT
- Works for arbitrary bounded domains
- Able to incorporate with mesh adaptation strategies
- Solvability guaranteed when $h_{\text{FD}} \le a_{\min}/(d+1)/\sqrt{d}$
- Effective preconditioners based on ILU and circulant matrix
- Numerical examples show: GoFD has convergence order $\mathcal{O}(h^{\min(1,0.5+s)})$ for quasi-uniform meshes and $\mathcal{O}(\bar{h}^2)$ for adaptive meshes, both in L^2 norm.

References

- J. Shen & WH, A grid-overlay finite difference method for the fractional Laplacian on arbitrary bounded domains, SIAM J. Sci. Comput., 46 (2024), A744-A769.
- J. Shen, B. Shi, & WH, Meshfree finite difference solution of homogeneous Dirichlet problems of the fractional Laplacian, Comm. Appl. Math. Comput., 7 (2025) 589-605.
- J. Shen & WH, Approximating and preconditioning the stiffness matrix in the GoFD approximation of the fractional Laplacian, Comm. Comput. Phys. 37 (2025), 1-29.
- J. Shen & WH, A grid-overlay finite difference method for inhomogeneous Dirichlet problems of the fractional Laplacian on arbitrary bounded domains, J. Sci. Comput. 102 (2025), 50.

Thank you for your attention!