Applied Differential Equation

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Chapter 1

Introduction

General introduction:

- What are DE?
- Where do they come from?
- How many types of them?
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  - Systems of 1st order linear DEs
  - Numerical approximation – Soln
- The Laplace transform
- Numerical approximation – Soln

1.1 Mathematical Modeling

Mathematical Model: a differential equation that describes some physical process.

Problem 1 Find the indefinite integral of function e^{2x}.

Everyone knows the soln $= \frac{1}{2}e^{2x} + C = \int e^{2x}dx$

Let $u = u(x)$ the indefinite integral (unknown function)

Definition: The indefinite integral of e^{2x} is the function whose derivative equals e^{2x}.

$$\boxed{\frac{du}{dx} = e^{2x}}$$

This is the mathematical model of the problem.

Soln:

$$u = \int e^{2x} dx = \frac{1}{2} e^{2x} + C$$

(Note: an infinite number of such solns)

Problem 2: The law of natural growth or decay

"The change rate of an amount of a radioactive substance, such as radium, is proportional to the amount at the current time."

$R(t)$: the amount of the substance at time t

$$\boxed{\frac{dR}{dt}} \quad \text{this is called Differential Equation}$$

where R is a constant depending on the material property of the substance.

Example: $k = 2$
$\frac{dR}{dt} = 2R$ then $R(t) = Ce^{2t}$, C: any number

Problem 3: A falling object

Consider an object that is falling in the atmosphere near sea level. Formulate a DE that describes the motion.

Newton's Second Law: $F = ma$.

F: force, m: mass, a: acceleration.
g: acceleration due to gravity $= 9.8m/sec^2$.
γ: drag coefficient (e.g. $\gamma = 2kg/sec$)
v: velocity.
$F = mg - \gamma v$, $ma = m\frac{dv}{dt}$

$$\boxed{m\frac{dv}{dt} = mg - \gamma v}$$

Example: $\gamma = 2kg/sec$, $m = 10kg$

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

Soln=?

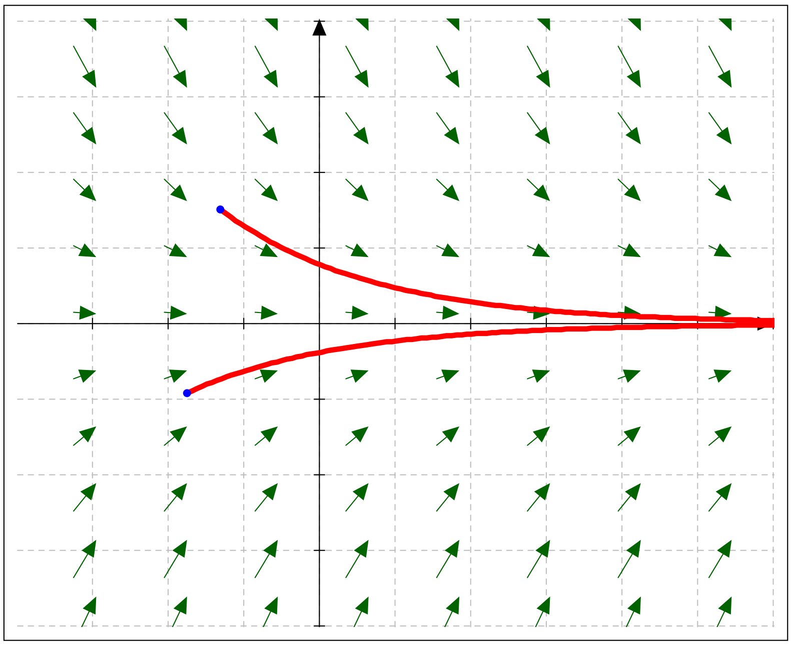
1.2 Direction fields

Example 1:

$$\frac{dy}{dt} = -y$$

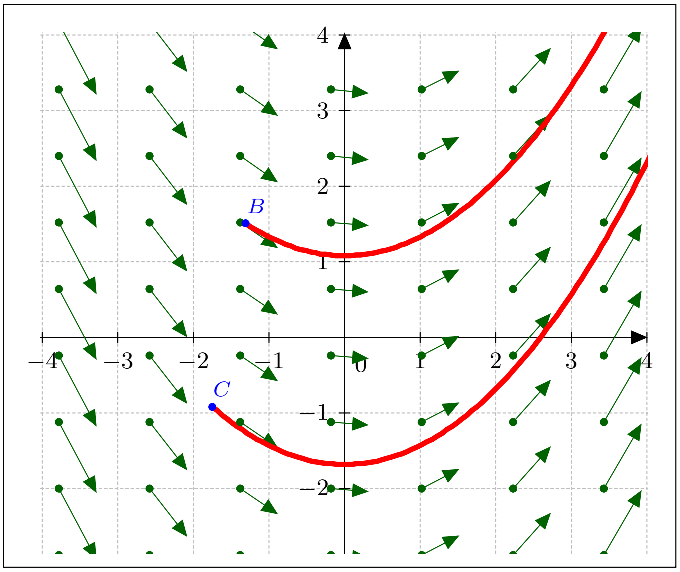
If $y = y(t)$ be the solution, then at the point (t_0, y_0), the slope of the tangent line to $y(t)$ is $-y_0$. Direction filed is used to described the slope of the tangent line to $y(t)$.

For any point on the t-y plane there will be a solution curve passing through it. Given a point on the plane, we can roughly draw the solution curve according to the slope field, for examples:



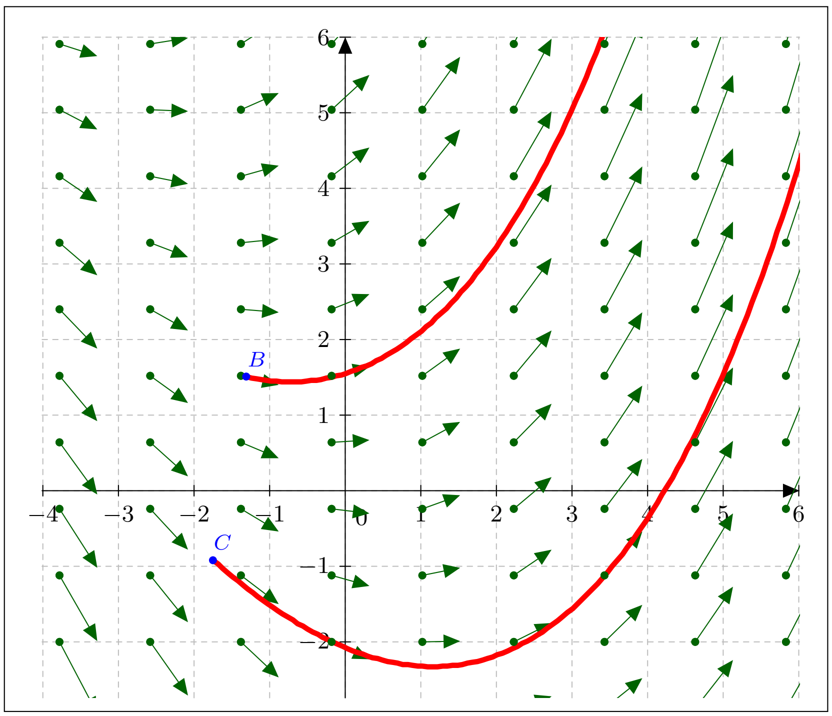
Example 2: $\dfrac{dy}{dx} = -\dfrac{x}{2}$.

We draw the slope field according to the following rule. At point (x_0, y_0), we draw a ray of slope $-\dfrac{x_0}{2}$.

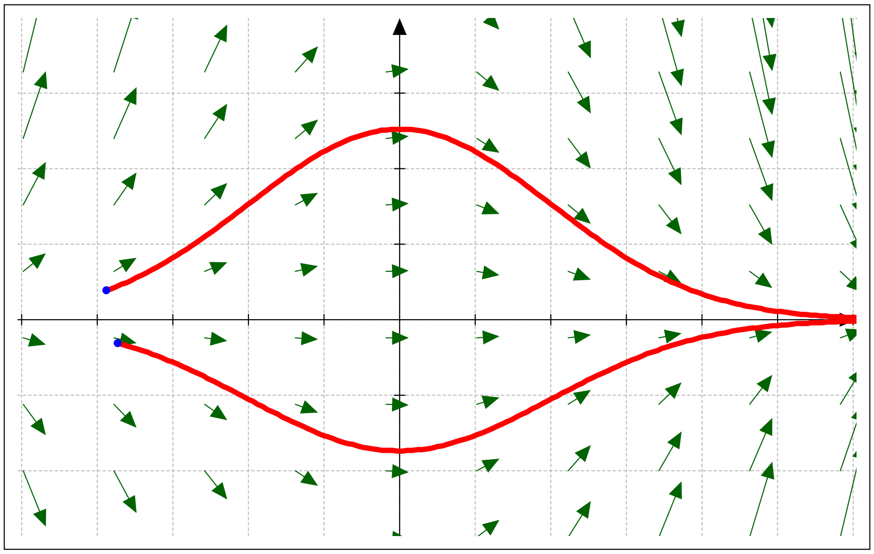


Example 3: $\dfrac{dy}{dx} = 0.4x + 0.2y$.

At point (x_0, y_0), we draw a ray of slope $0.4x_0 + 0.2y_0$.



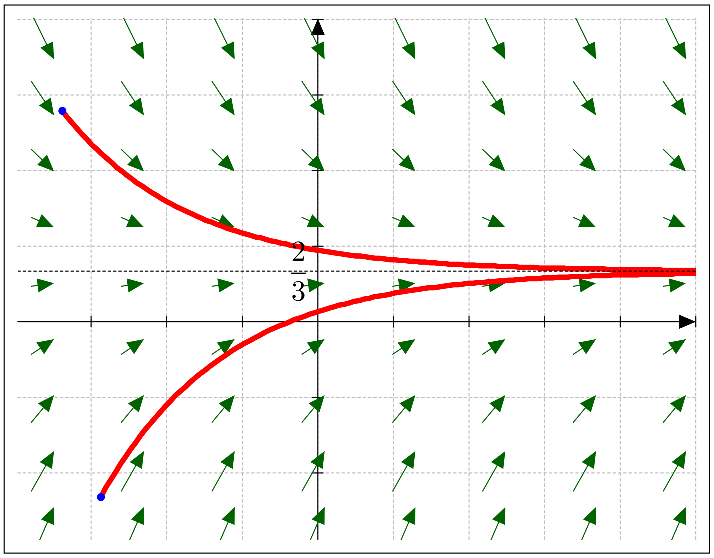
Example 4: $\dfrac{dy}{dx} = -\dfrac{xy}{4}$.



Example 5: For a DE which can be written in a form $\dfrac{dy}{dt} = ay + b$, for example $\dfrac{dy}{dt} = -3y + 2$,whose solution have the required behavior as $t \to \infty$, all solutions approach $y = \dfrac{2}{3}$.

$$\frac{dy}{dt} = -3y + 2$$
$$\frac{dy}{-3y+2} = dt$$
$$\int \frac{1}{-3y+2}dy = \int 1dt$$
$$\int \frac{1}{y}dy = \ln|y| \Rightarrow \int \frac{1}{-3y+2}dy = -\frac{1}{3}\ln|-3y+2|$$
$$-\frac{1}{3}\ln|-3y+2| = t + C, \quad \ln|-3y+2| = -3t + C$$
$$|-3y+2| = e^{-3t+C} = Ce^{-3t}, \quad -3y + 2 = Ce^{-3t}$$
$$y = Ce^{-3t} + \frac{2}{3}$$

Since $\lim\limits_{t\to\infty} e^{-3t} = 0$, $\lim\limits_{t\to\infty} y = \dfrac{2}{3}$.



1.3 Classification of DEs

<u>**Definition of DE:**</u> Suppose that there is an independent variable (say t) and there is a dependent variable that is an unknown function of t (say $y(t)$). Then a DE is an identity that relates the independent variable, dependent variable, and its derivative.

Examples

- $\frac{dy}{dt} = y$, (1st order)
- $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = t$, (2nd order)
- $\frac{dy}{dt} = \sqrt{y^2 + 1}$, (1st order)

Orders: order of the highest derivative appearing in the DE.

Solution of a DE: a function satisfying the equation identically.

General Solution vs particular Solution: The general solution is a form of function such that every solution of the DE can be cast in the form.

Example $\frac{dy}{dt} = -y$

- (particular) Solutions: $y = e^{-x}$ (verify)

- (General solution) $y = Ce^{-x}$, C any real number.

Integral Curve=graph of a (particular) solution

General solution=a family of integral curves.

Linear and non-linear eqns:

$$F(t, y, y', ..., y^{(n)}) = 0$$

is said to be linear if F is a linear function of the variables y,y',...,y^n. Otherwise, it is a non-linear equation.

The general linear ordinary DE of order n is

$$\boxed{a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t)}$$

Example: 1st order linear DE:

$$y' + p(t)y = g(t)$$

Example 1: Order and linearity

- $(1+y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$, 2nd order, non-linear
- $\dfrac{dy}{dt} + ty^2 = 0$, first order, non-linear.
- $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (\cos^2 t)y = t^3$, 3rd order, linear.
- $\sqrt{1+y'} = t$, first order, non-linear.
- $\dfrac{1+t^2y'}{1+t^3y''} = 2t$, second order, linear.
- $\dfrac{1+t^2y'}{1+t^3y''} = 2t + y$, second order, nonlinear.

Example 2: Determine the value of r such that $y = e^{rt}$ is a soln of

$$y'' + y' - 6y = 0$$

Soln: Replace y by e^{rt} on the right hand-side,

$$(e^{rt})'' + (e^{rt})' - 6e^{rt} = r^2(e^{rt}) + r(e^{rt}) - 6e^{rt} = (r^2 + r - 6)e^{rt} = 0.$$

Therefore, it is equivalent to solve

$$\begin{aligned} &r^2 + r - 6 = 0 \\ &(r+3)(r-2) = 0 \\ &r = -3, \ r = 2 \end{aligned}$$

Chapter 2

First Order Differential Equations

$$y' = f(t, y) \text{ or } \frac{dy}{dt} = f(t, y)$$

Main task: find the general solution.

2.1 Linear equations

The simplest case: $f(t, y)$ is independent of y

$$\boxed{y' = f(t)} \quad \boxed{y = \int f(t)dt}$$

Example 1 $y' = \cos t$

$$y = \int \cos t dt = \sin t + C$$

Linear 1st order DEs:

$f(t, y)$ is a linear function about y. For example

$$f(t, y) = -p(t)y + g(t)$$

where $p(t)$ and $g(t)$ are given functions.

$$\boxed{y' + p(t)y = g(t)}$$

Solution method – integration factor

Example 2: $y' - 2ty = t$, so $p(t) = -2t$, $g(t) = t$

Solution: key – rewrite the DE into a form that can be solved easily

$\mu = \mu(t)$: A function to be determined

$\mu y' - 2t\mu y = \mu t$

$$\underbrace{\mu y' + \mu' y}_{[\mu y]'} - \mu' y - 2t\mu y = \mu t$$

$$[\mu y]' + \underbrace{[-\mu' - 2t\mu]}_{\text{choose } \mu \text{ such that this is zero}} y = \mu t$$

$[\mu y]' = \mu t$

$$\mu y = \int \mu(t) t dt + C$$

$$y = \frac{\int \mu(t) t dt + C}{\mu(t)}$$

Find $\mu(t)$:

$-\mu' - 2t\mu = 0$

$\mu' = -2t\mu$ or $\dfrac{d\mu}{dt} = -2t\mu$

$$\frac{d\mu}{\mu} = -2t dt \quad \int \frac{1}{\mu} d\mu = \int -2t dt$$

$\ln \mu = -t^2 + C$

$\mu(t) = e^{-t^2} e^C$

$\mu(t) = e^{-t^2}$ (Choose a simple one!)

so

$$y = \frac{\int e^{-t^2} \cdot t dt + C}{e^{-t^2}} = e^{t^2}\left[-\frac{1}{2}e^{-t^2} + C\right]$$

$$\boxed{y = -\frac{1}{2} + Ce^{t^2}}$$

General Case:

$$\boxed{\begin{gathered} y' + p(t)y = g(t) \\ y = \frac{\int \mu(t) g(t) dt + C}{\mu(t)} \\ \mu(t) = e^{\int p(t) dt} \end{gathered}}$$

Proof: $y' + p(t)y = g(t)$. Multiply $\mu = \mu(t)$ to both sides.

$$\mu y' + \mu p(t)y = \mu g(t) \quad \mu \text{ to be determined}$$

$$\mu y' + \mu' y - \mu' y + \mu p(t)y = \mu g(t)$$

$$\underbrace{\mu y' + \mu' y}_{=(\mu y)'} - \underbrace{\mu' y - \mu p(t)y}_{\text{choose } \mu \text{ such that this is zero}} = \mu g(t)$$

$$\mu' y - \mu p(t)y = 0 \Rightarrow \mu' = \mu p(t) \quad \frac{\mu'}{\mu} = p(t)$$

$$\int \frac{\mu' dt}{\mu} = \int p(t)dt \quad \int \frac{d\mu}{u} = \int p(t)dt \quad \ln \mu = \int p(t)dt$$

$$\boxed{\mu = e^{\int p(t)dt}}$$ we just need to pick one particular μ, so we do not put C, the constant term here.

$$(\mu y)' = \mu g(t)$$

$$\mu y = \int \mu(t)g(t)dt + C$$

$$\boxed{y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)}}$$

Examples:

1. $y' + \frac{1}{2}y = 2\cos t$

   $p(t) = \frac{1}{2}$, $g(t) = 2\cos t$

   $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{1}{2}dt} = e^{t/2}$

   $$\int \mu(t)g(t)dt = \int e^{t/2} \cdot 2\cos t dt = \frac{4e^{t/2}\cos t + 8e^{t/2}\sin t}{5}$$

   $$y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{2e^{t/2}\sin t - 4e^{t/2}\cos t + C}{5e^{t/2}} = \frac{4\cos t + 8\sin t}{5} + Ce^{-\frac{t}{2}}$$

   (Note: $Ce^{-\frac{t}{2}}$ is a general solution to $y' + \frac{1}{2}y = 0$.)

   Verification:

   $$\frac{d}{dt}\left(\frac{4\cos t + 8\sin t}{5}\right) + \frac{1}{2}\left(\frac{4\cos t + 8\sin t}{5}\right) = \frac{-4\sin t + 8\cos t}{5} + \frac{2\cos t + 4\sin t}{5} = 2\cos t$$

2. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}$

   $p(t) = \frac{2}{t}$, $g(t) = \frac{\cos t}{t^2}$

   $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$

$$\int \mu(t)g(t)dt = \int t^2 \cdot \frac{\cos t}{t^2}dt = \int \cos t dt = \sin t$$
$$y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{\sin t + C}{t^2}$$

Initial Value Problem (IVP):

Sometimes it is important to pick out one particular solution. This is done by specifying and auxiliary condition (initial condition)

$$y(t_0) = y_0$$

or specifying that the solution curve should pass through (t_0, y_0)

DE + initial condition = IVP

Example 2 Solve

$$y' - 2y = 4 - t$$

Sketch the graphs of several solutions. Find the initial point on the y-axis that separates solutions that grow large positively from those that grow large negatively as $t \to \infty$

Solution $p(t) = -2$, $g(t) = 4 - t$

$$\mu(t) = e^{\int p(t)dt} = e^{-2t}$$

$$\begin{aligned} y &= \frac{\int \mu(t)g(t)dt + C}{\mu(t)} \\ &= \frac{\int e^{-2t}(4-t)dt + C}{e^{-2t}} \\ &= e^{2t}[\int e^{-2t}(4-t)dt - \int e^{-2t}tdt + C] \\ &= e^{2t}\left[-2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + C\right] \\ &= -2 + \frac{1}{2}t + \frac{1}{4} + Ce^{2t} \end{aligned}$$

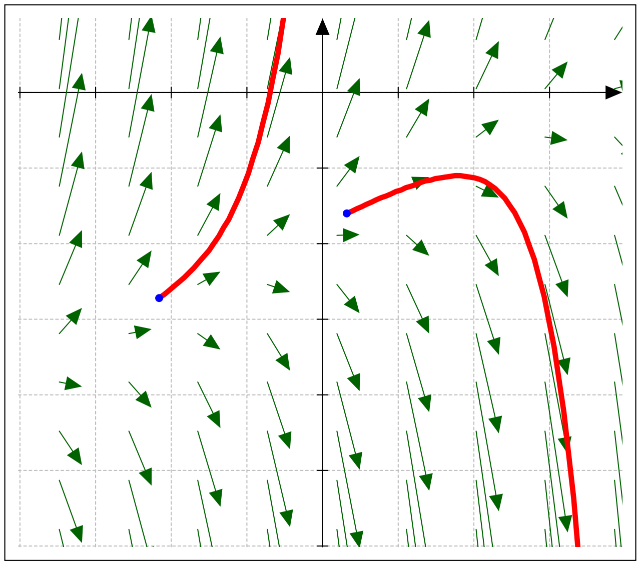
$$\begin{aligned} \int e^{-2t}tdt &= -\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}dt \\ &= -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{2t} \end{aligned}$$

$$\boxed{y = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}}$$

- $C > 0$: $\lim_{t\to\infty} y = \infty$
- $C < 0$: $\lim_{t\to\infty} y = -\infty$

$$C = \frac{y + \frac{7}{4} - \frac{1}{2}t}{e^{2t}}$$

The threshold is the straight line $y = -\frac{1}{2}t - \frac{7}{4}$



Example 3: Find the solution of IVP $ty' + 2y = \sin t, \quad y\left(\frac{\pi}{2}\right) = 1$

Soln: $y' + \frac{2y}{t} = \frac{\sin t}{t}$.

$p(t) = \frac{2}{t}, \quad g(t) = \frac{\sin t}{t}$

$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = t^2$

$$
\begin{aligned}
y &= \frac{\int t^2 \frac{\sin t}{t} dt + C}{t^2} \\
&= t^{-2} \left(\int t \sin t dt + C \right), \quad \sin t dt = d(-\cos t) \\
&= t^{-2} \left(\int t d(-\cos t) + C \right), \quad \int u dv = uv - \int v du \\
&= t^{-2} (-t \cos t + \int \cos t dt + C)
\end{aligned}
$$

$y = t^{-2}(-t \cos t + \sin t + C)$

$y\left(\frac{\pi}{2}\right) = 1, \quad 1 = \frac{1}{\left(\frac{\pi}{2}\right)^2} \left(-\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} + C \right)$

$\left(\frac{\pi}{2}\right)^2 = 1 + C$

$C = \left(\frac{\pi}{2}\right)^2 - 1$

$$\boxed{y = t^{-2}\left[-t\cos t + \sin t + \left(\frac{\pi}{2}\right)^2 - 1\right]}$$

Methods of Variation of parameters:

$$y' + p(t)y = g(t) \quad \text{inhomogeneous eqn.}$$

homogeneous equation (every term involves either y or y')

$$\begin{aligned}
&y' + p(t)y = 0\\
&\frac{y'}{y} = -p(t)\\
&[\ln y]' = -p(t)\\
&\ln y = -\int p(t)dt + C_1\\
&\boxed{y = Ce^{-\int p(t)dt}} \quad \text{Solution of the homogeneous eqn.}
\end{aligned}$$

To find the solution of the inhomogeneous eqn, let

$$\begin{aligned}
&y = A(t)e^{-\int p(t)dt}\\
&A'(t)e^{-\int p(t)dt} + A(t)e^{-\int p(t)dt}[-p(t)] + p(t)A(t)e^{-\int p(t)dt} = g(t)\\
&A'(t) = g(t)e^{\int p(t)dt}
\end{aligned}$$

Denote

$$\mu(t) = e^{\int p(t)dt}$$

$$A'(t) = g(t)\mu(t) \text{ or } A(t) = \int \mu(t)g(t)dt + C$$

$$\boxed{y = \frac{A(t)}{\mu(t)} = \frac{\int \mu(t)g(t)dt + C}{\mu(t)}}$$

2.2 Separable equations

$$y' = f(t,y), \quad f(t,y) = -\frac{M(t)}{N(y)}$$

$f(t,y)$ is a product of two functions, one of them is a function of only t and the other is a function of only

y.

$$\frac{dy}{dt} = -\frac{M(t)}{N(y)}$$

$$N(y)dy = -M(t)dt \quad \text{or} \quad \boxed{M(y)dy + N(t)dt = 0}$$

$$\underbrace{\int N(y)dy}_{H_2(y)} = -\underbrace{\int M(t)dt}_{H_1(t)}$$

$$H_2(y) = -H_1(t) + C$$

$$H_1(t) + H_2(y) = C \quad \text{implicit solution}$$

Solution: if you can simplify it, simplify. If not, this is the solution – implicit solution.

Example 1: $\dfrac{dy}{dt} = -3y + 2$

$$\frac{dy}{dt} = -3y + 2$$

$$\frac{dy}{-3y+2} = dt$$

$$\int \frac{1}{-3y+2}dy = \int 1dt$$

$$\int \frac{1}{y}dy = \ln y \ \Rightarrow \int \frac{1}{-3y+2}dy = -\frac{1}{3}\ln -3y + 2$$

$$-\frac{1}{3}\ln(-3y+2) = t + C, \quad \ln(-3y+2) = -3t + C$$

$$|-3y+2| = e^{-3t+C} = Ce^{-3t}, \quad -3y + 2 = Ce^{-3t}$$

$$y = Ce^{-3t} + \frac{2}{3}$$

Since $\lim_{t\to\infty} e^{-3t} = 0$, $\lim_{t\to\infty} y = \dfrac{2}{3}$.

$$\boxed{\begin{gathered} y' = ay + b \text{ or } \frac{dy}{dt} = ay + b \\ y = Ce^{at} - \frac{b}{a} \end{gathered}}$$

Example 1: $\frac{dy}{dt} = \frac{t - e^{-t}}{y + e^y}, \quad y(0) = 1$

$$\begin{aligned}
&(y + e^y)dy = (t - e^{-t})dt \\
&\int (y + e^y)dy = \int (t - e^{-t})dt \\
&\frac{y^2}{2} + e^y = \frac{t^2}{2} + e^{-t} + C \\
&\frac{y^2}{2} + e^y - \frac{t^2}{2} - e^{-t} = C
\end{aligned}$$

$y(0) = 1$, $\frac{1^2}{2} + e^1 - \frac{0^2}{2} - e^0 = C$

$C = \frac{1}{2} + e - 1 = e - \frac{1}{2}$.

Example 2: $\begin{cases} y' = 1 + x + y^2 + xy^2 = (1 + x) + y^2(1 + x) = (1 + y^2)(1 + x) \\ y(0) = 0 \end{cases}$

$$\begin{aligned}
&\text{Fact:} \quad \int \frac{1}{1 + y^2} dy = \arctan y \\
&\int \frac{1}{1 + y^2} dy = \int 1 + x dx \\
&\arctan y = x + \frac{x^2}{2} + C \\
&\text{find C: } \arctan(0) = 0 + \frac{0^2}{2} + C \\
&C = 0 \\
&\arctan y = x + \frac{x^2}{2} \text{ or } \boxed{y = \tan(x + \frac{x^2}{2})}
\end{aligned}$$

Example 3: $\begin{cases} y' = \frac{1 + 3x^2}{3y^2 - 6y} \\ y(0) = 1 \end{cases}$

Example 4: $y' = \frac{2y + 1}{y - 1}$

2.3 Mathematical Modeling with 1st Order equations

Recall The general solution of $y' = ay + b$, $a \neq 0$.

$$y' = ay + b \text{ or } \frac{dy}{dt} = ay + b$$

$$y = Ce^{at} - \frac{b}{a}$$

Example 1(Water tank problem)

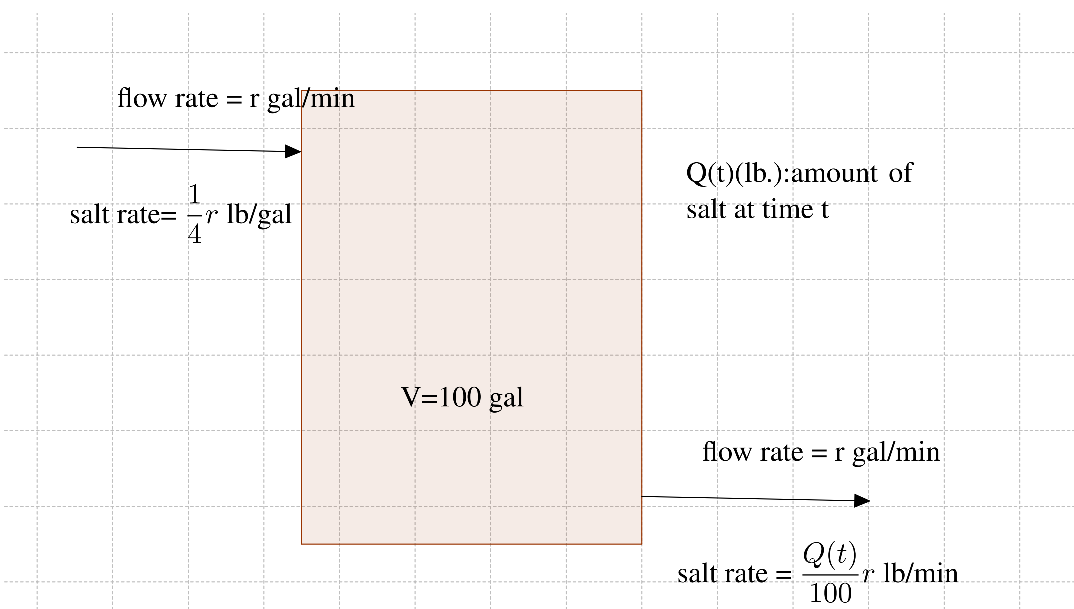
At time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing $\frac{1}{4}$ (lb of salt)/gal is entering the tank at a rate of r gal/min, and that the well-stirred mixture is draining from the tank at the same rate.

(a)Set up the IVP describing this flow process.

(b)Find the amount of salt $Q(t)$ in the tank at any time; and also find the limiting amount Q_L that is present after a very long time.

(c)If $r = 3$, and $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L.

(d)Find the flow rate that is required if the value of T is not to exceed 45 min.



- t: in min

- $Q(t)$: amount of salt in the tank
- r_{in}: rate of salt poured in the tank.
  r_{in} = incoming density of salt $\times$ incoming rate of solution $= \dfrac{1}{4}r$
- $\dfrac{Q(t)}{100}$: density of salt in the tank at time t.
- $r_{out} = \dfrac{Q}{100} \cdot r$: rate of salt following out.
- $Q(0) = Q_0$: initial condition
- (a) $\begin{cases} \dfrac{dQ}{dt} = r_{in} - r_{out} = \dfrac{r}{4} - \dfrac{Qr}{100} \\ Q(0) = Q_0 \end{cases}$
- $a = -\dfrac{r}{100}, \quad b = \dfrac{r}{4}, \quad \dfrac{b}{a} = -25 \; Q(t) = Ce^{-(r/100)t} + 25$
- (b) $Q(0) = Q_0, \quad C + 25 = Q_0, \quad \boxed{Q(t) = (Q_0 - 25)e^{-(r/100)t} + 25}$
  $Q_L = \lim_{t\to\infty} (Q_0 - 25)e^{-(r/100)t} + 25 = 25$
- (c) $Q(t) = (2Q_L - Q_L)e^{-3/100t} + Q_L = 1.02Q_L$
  $e^{-0.03t} + 1 = 1.02, \quad -0.03t = \ln 0.02, \quad t = 130.4$ min
- (d) $Q(45) = (2Q_L - Q_L)e^{-r/100\times 45} + Q_L = 1.02Q_L$
  $e^{-0.45r} + 1 = 1.02, \quad -0.45r = \ln 0.02, \quad r = 8.69$ lb/gal

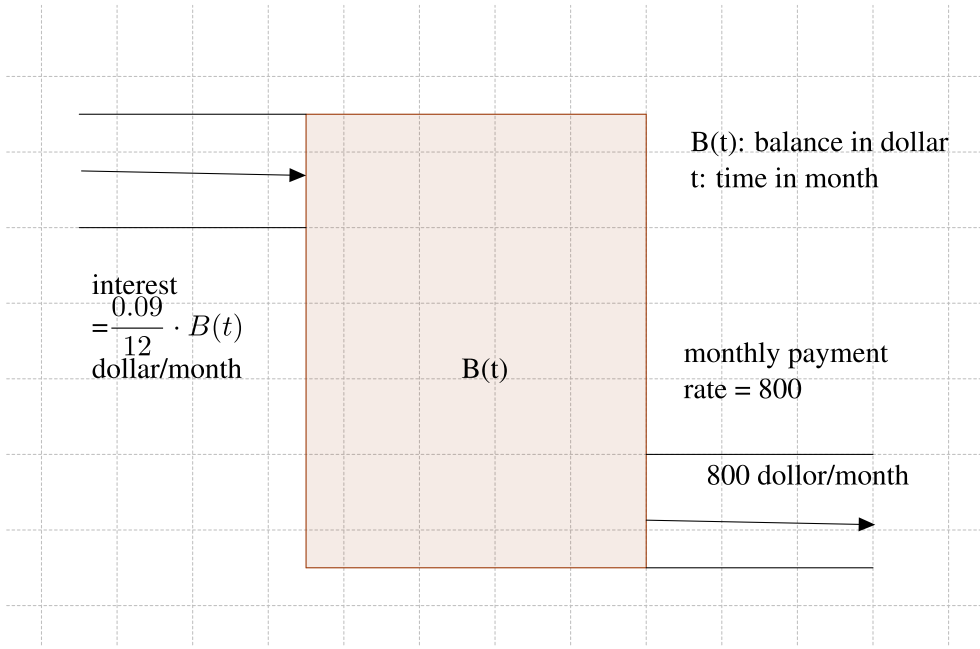
Example 2(Mortgage Problem)

A home buyer can afford to spend no more than $\$800/month$ on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 30 years.

Assume that interest is compounded continuously and that payments are also made continuously.

(a) Determine the maximum amount that this buyer can afford to borrow.

(b) Determine the total interest paid during the term of the mortgage.



- t: in month
- $B(t)$: balance (money owed to the bank)
- monthly interest rate$= \frac{0.09}{12} = 0.0075$
- $r_{in} = 0.0075B(t)$: increment of balance per month = interest per month
- $r_{out} = 800$: decrement of balance per month = monthly payment.
- $B(0) = B_0$: initial amount of money borrowed from the bank.
- $\begin{cases} \frac{dB}{dt} = r_{in} - r_{out} = 0.0075B - 800 \\ B(0) = B_0 \end{cases}$
- $B(t) = Ce^{0.0075t} + \frac{800}{0.0075}$
- $B(0) = C + \frac{800}{0.0075} = B_0$, $\boxed{B(t) = \left(B_0 - \frac{800}{0.0075}\right) e^{0.0075t} + \frac{800}{0.0075}}$
- (a) Find B_0. $B(30 \times 12) = 0$, $\left(B_0 - \frac{800}{0.0075}\right) e^{0.0075 \times 30 \times 12} + \frac{800}{0.0075} = 0$

  $B_0 = 99,498$
- (b) Find the total interest

Method 1: $TI = \int_0^{30\times 12} 0.0075B(t)dt$

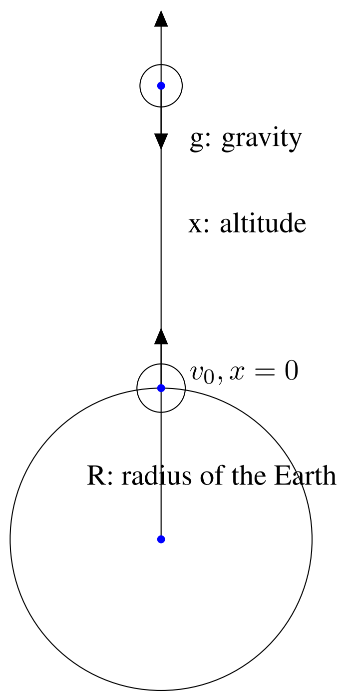
Method 2: $TI =$ total payment $- Q_0 = 800 \times 30 \times 12 - Q_0 = 288,000 - 99,498$

Example 3 (Escape Velocity)

A body of constant mass m is projected away from the earth in a direction perpendicular to the Earth's surface with an initial velocity v_0. The gravitational force acting on the body is inversely proportional to the square of the distance from the center of the earth and is given by $w(x) = k/(x+R)^2$, where R is the radius of the earth, and x is the distance between the body and the surface of the earth Assuming that there is no air resistance, but taking into account the variation of the Earth's gravitational field with distance,

(a)find an expression for the velocity during the ensuing motion,

(b)find the escape velocity.



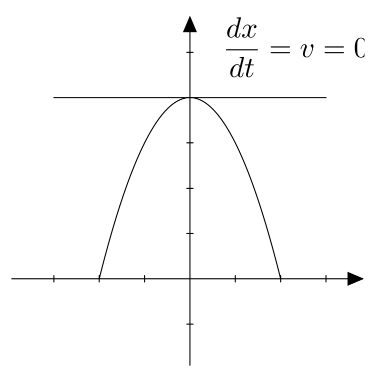
Soln:

- $F = ma, \quad a = \dfrac{dv}{dt}.$
- t in second

- $x(t)$: distance between the body and the surface of the earth at time t.
- R: radius of the earth.
- $m\frac{dv}{dt} = -w = -\frac{k}{(x+R)^2}$: negative sign signifies that w is directed in the negative v direction (starts from the surface and goes up).
- $w = mg = \frac{k}{(0+R)^2}, \quad k = mgR^2$: on the surface of earth ($x = 0$), $w = mg$.
- $\frac{dv}{dt} = \frac{dv}{dx} \cdot \underbrace{\frac{dx}{dt}}_{=v} = v\frac{dv}{dx}$: reduce variable t. From now on, v is a function of x.
- $v(0) = v_0$: on the surface of the earth ($x = 0$), velocity is v_0.
- (a) $\begin{cases} mv\frac{dv}{dx} = -\frac{mgR^2}{(x+R)^2} \\ v(0) = v_0 \end{cases}$

  $$vdv = -\frac{gR^2}{(x+R)^2}dx$$
  $$\int vdv = -gR^2 \int \frac{1}{(x+R)^2}dx$$
  $$\frac{1}{2}v^2 = \frac{gR^2}{x+R} + C$$
  $$v(0) = v_0, \quad \frac{1}{2}v_0^2 = \frac{gR^2}{R} + C, \quad C = \frac{1}{2}v_0^2 - gR$$
  $$\boxed{v(t) = \pm\sqrt{\frac{2gR^2}{x+R} + v_0^2 - 2gR}}$$

- (b) Find the escape velocity.



Set $v = 0$, maximize altitude $x_{max} = \frac{v_0^2 R}{2gR - v_0^2}$ or $v_0 = \sqrt{2gR \cdot \frac{x_{max}}{R + x_{max}}}$.
Let $x_{max} \to \infty$ (escape !) $\Rightarrow v_0 = \sqrt{2gR}$ (escape velocity).
$g = 9.8m/s^2$, $R = 6,286$ km, $\boxed{v_\infty \sim 11.1km/sec}$

2.5 Autonomous Equations and Population Dynamics

Autonomous equations: the independent variable (in the following case t) does not appear explicitly.

$$\frac{dy}{dt} = f(y)$$

has an important application in population dynamics.

objective: study a geometric method to obtain important qualitative information directly from the differential equation, without solving the equation — qualitative analysis.

- $y(t)$: population at time t.
- r: growth rate of the population.

1. Exponential Growth: (e.g.,when the resource is unlimited, such that the growth is roughly constant.)

Ex 1. In average, one rabbit reproduces r new baby rabbits per year. Totally, the population of rabbit increases by ry per year.

$$\begin{cases} \frac{dy}{dt} = ry, \quad r > 0: \text{ growth rate} \\ y(0) = y_0 \end{cases}$$

Soln:

$$\frac{dy}{y} = rdt$$

$$\int \frac{dy}{y} = \int rdt + C$$

$$\ln y = rt + C$$

$$y = e^{rt+C} = e^C \cdot e^{rt} = Ce^{rt}, \quad \text{general solution}$$

$$y(0) = Ce^0 = y_0, \quad C = y_0$$

$$\boxed{y = y_0 e^{rt}}$$

Ex 2. In average, one rabbit reproduces r new baby rabbits per year. And at the same time, 10% rabbit dies per year (mortality rate is 0.1 per year). The population of rabbits changes by $ry - 0.1y$ per year.

$$\begin{cases} \frac{dy}{dt} = (r - 0.1)y \\ y(0) = y_0 \end{cases}$$

Soln: $\boxed{y = y_0 e^{(r-0.1)t}}$

Summary: When the growth rate is constant, then it's a exponential growth model. $\lim_{t\to\infty} y(t) = \infty$.

2. Logistic Growth: (e.g., when resource is limited) grow rate depends on the population or $r = h(y)$, such that

- $h(y) \simeq r$, when y is small.
- $h(y) \searrow$, when $y \nearrow$.
- $h(y) < 0$, when y is large.

$\Rightarrow\ h(y) = r - ay = r\left(1 - \frac{y}{k}\right)$.

$\Rightarrow$ Logistic equation:

$$\boxed{\frac{dy}{dt} = r\left(1 - \frac{y}{k}\right) y}$$

r: intrinsic growth rate.

k: saturation level.

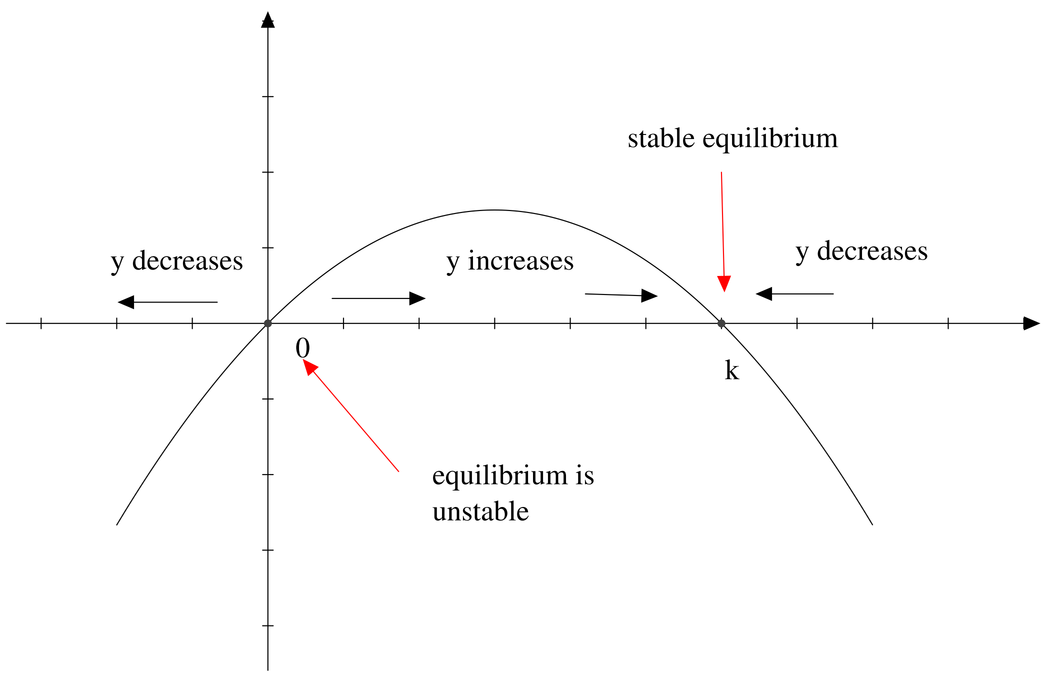
Qualitative analysis of the logistic equation:

Let $f(y) \equiv r\left(1 - \frac{y}{k}\right) y$

- Step 1: Set $f(y) = 0$, obtain zeros: $y = 0$ and $y = k \Rightarrow$ two constant solutions:

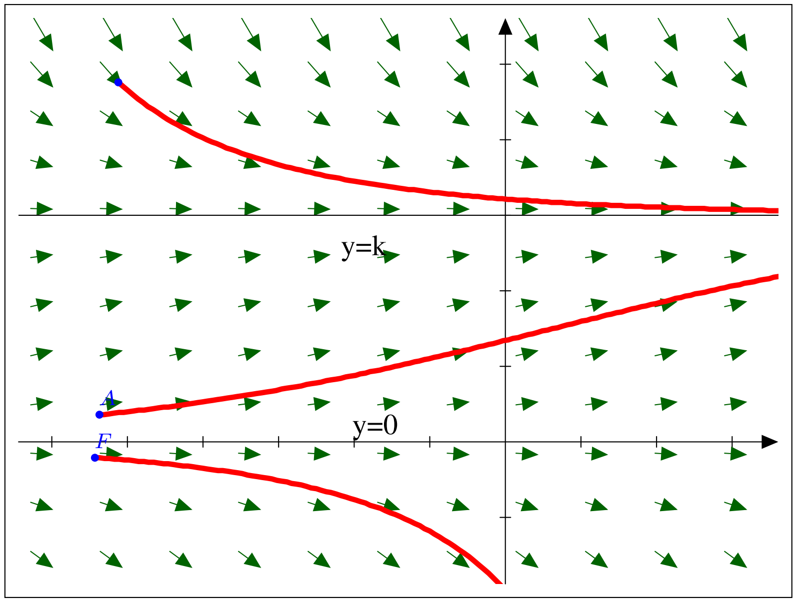
$$y = \phi_1(t) = 0, \quad y = \phi_2(t) = k.$$

- Step 2: sketch the graph of $f(y)$:



- $\frac{dy}{dt} > 0$, for $0 < y < k$
- $\frac{dy}{dt} < 0$, for $y > k$
- $\frac{dy}{dt} < 0$, for $y < 0$

- Step 3: sketch integral curves



$\phi_1(t) = k$: asymptotically stable because every nearby integral curve is converging to k as $t \to \infty$.

$\phi_2(t) = 0$: asymptotically unstable because every nearby integral curve is leaving. A little disturbing will drag away the curve.

Verifying (Separable equation) $\frac{dy}{\left(1 - \frac{y}{k}\right) y} = r dt$

partial fraction $\frac{1}{\left(1 - \frac{y}{k}\right) y} = \frac{1}{y} + \frac{1}{k - y}$

$$\int \frac{1}{y} + \frac{1}{k-y} dy = \int r dt + C$$

$$\ln y - \ln(k - y) = rt + C, \quad \underbrace{\boxed{\ln\left(\frac{y}{k-y}\right) = rt + C}}_{\text{Logistic Model}}, \quad \frac{y}{k-y} = C_2 e^{rt}$$

$$y(0) = y_0, \quad \frac{y_0}{k - y_0} = C_2$$

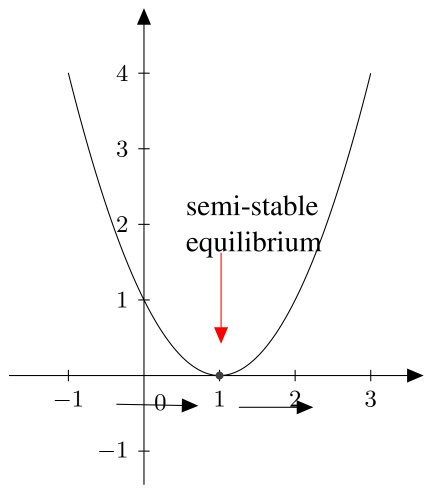
$\Rightarrow y = \dfrac{y_0 k}{y_0 + (k - y_0)e^{-rt}}$

$t \to \infty, y \to k.$

Example 1: (Semi-stable) $\dfrac{dy}{dt} = (1-y)^2$

Zeros: $y = 1$.

Graph of $(1-y)^2$



Example 2 $\dfrac{dy}{dt} = e^y - 1$

zeros: $y = 0$

signs of $f(y) = e^y - 1$.

- $y < 0$, $f(y) < 0$
- $y > 0$, $f(y) > 0$

$y = 0$ is a non-stable equilibrium.

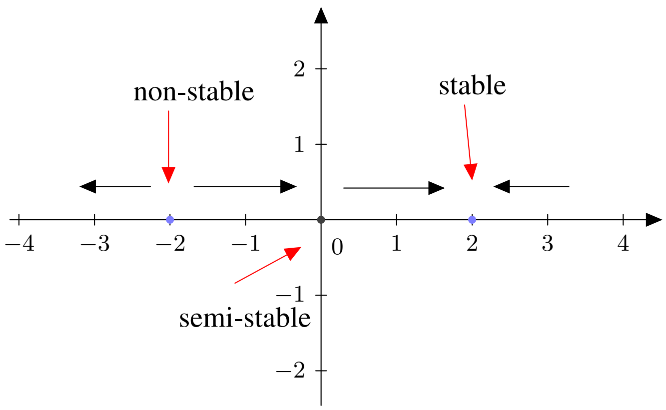
Example 3 $\dfrac{dy}{dt} = y^2(4 - y^2)$

zeros: $y = 0$, $y = -2$, $y = 2$

signs of $f(y) = y^2(4 - y^2)$.

- $y < -2$, for example $f(-3) = 9(4-9) < 0$

- $-2 < y < 0$, $f(-1) = 1(4-1) > 0$
- $0 < y < 2$, $f(1) = 1(4-1) > 0$
- $y > 2$, $f(3) = 9(4-9) < 0$



2.6 Exact Equations

$$\frac{dy}{dt} = f(t,y) \equiv -\frac{M(t,y)}{N(t,y)}$$

1. Rewrite the DE into

$$\boxed{M(t,y)dt + N(t,y)dy = 0}$$

2. Consider the total variation of $\psi(t,y)$:

$$\boxed{d\psi(t,y) \equiv \frac{\partial \psi}{\partial t}dt + \frac{\partial \psi}{\partial y}dy}$$

which measures the change of ψ when t and y undergo a small change.

If $\frac{\partial \psi}{\partial t} = M(t,y)$, $\frac{\partial \psi}{\partial y} = N(t,y)$ then $d\psi(t,y) = M(t,y)dt + N(t,y)dt = 0 \Rightarrow \psi(t,y) = C$ (constant)

Condition for the existence of such a $\psi(t,y)$ satisfying $\frac{\partial \psi}{\partial t} = M(t,y)$, $\frac{\partial \psi}{\partial y} = N(t,y)$:

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial y}\right)$$

or

$$\boxed{\frac{\partial}{\partial y}M(t,y) = \frac{\partial}{\partial t}N(t,y)}$$

Theorem Rewrite the DE into

$$\underbrace{M(t,y)}_{\frac{\partial\psi}{\partial t}}\,dt + \underbrace{N(t,y)}_{\frac{\partial\psi}{\partial y}}\,dy = 0$$

If M and N satisfy

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

then there exists a function $\psi = \psi(t,y)$ such that

$$\frac{\partial\psi}{\partial t} = M(t,y), \quad \frac{\partial\psi}{\partial y} = N(t,y).$$

Definition of exact equation:

- How to find ψ.

  Examples $(y+4t) + (t-y)\dfrac{dy}{dt} = 0.$
  $M(t,y) = y+4t$, $N(t,y) = t-y$

$$\begin{cases} \psi(t,y) = \int M(t,y)dt = \int (y+4t)dt = yt + 2t^2 + C_1(y) & (1) \\ \psi(t,y) = \int N(t,y)dy = \int t - y dy = ty - \frac{y^2}{2} + C_2(t) & (2) \end{cases}$$

  So $yt + 2t^2 + C_1(y) = ty - \dfrac{y^2}{2} + C_2(t)$.

$$C_2(t) = 2t^2, \quad C_1(y) = -\frac{y^2}{2}$$

  So $\psi(t,y) = yt + 2t^2 - \dfrac{y^2}{2}$. The solution of the DE is

$$\psi(t,y) = yt + 2t^2 - \frac{y^2}{2} = C$$

  which is a implicit solution of $y(t)$. If given initial condition $y(1) = 0$, then

$$\psi(1,0) = 0 + 2 = C, \Rightarrow yt + 2t^2 - \frac{y^2}{2} = 2.$$

- verifying the solution.

$$\begin{cases} \frac{\partial}{\partial y}(yt + 2t^2 - \frac{y^2}{2}) = t - y = N(t,y) \\ \frac{\partial}{\partial t}(yt + 2t^2 - \frac{y^2}{2}) = y + 4t = M(t,y) \end{cases}$$

2.7 Theory of 1st Order DEs and difference between linear and non-linear DEs

For a given IVP

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

most concerned:existence and uniqueness of the solutions.

Theorem If functions $p(t)$ and $g(t)$ are continuous on an open interval $I : \alpha < t < \beta$ containing point $t = t_0$, then there exists a unique solution $y = \psi(t)$ of the IVP

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

$y = \psi(t)$ is valid for all $t \in I$.

Example: $y' + \frac{2}{t}y = \frac{\sin t}{t}$, $y\left(\frac{\pi}{2}\right) = 1$

$p(t) = \frac{2}{t}$, $g(t) = \frac{\sin t}{t}$ are continuous in $(-\infty, 0) \cup (0, \infty)$. But $t = \frac{\pi}{2}$ is in $(0, \infty)$.

Therefore there exists a unique solution in $t \in (0, \infty)$.

Theorem 2 (General case including nonlinear DEs)

If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R : \ |t| \leq a, \ |y| \leq b$, then there is some interval $|t| \leq h \leq a$ (Note: h can be very small) in which there exists a unique solution $y = \psi(t)$ of the IVP

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

Example 2 $\begin{cases} y' = y^2 \\ y(t_0) = y_0 \end{cases}$

$f(t,y) = y^2$, $\frac{\partial f}{\partial y} = 2y$ very smooth. But

$$y(t) = -\frac{1}{(t - t_0) - \frac{1}{y_0}} = \frac{1}{\frac{1}{y_0} - (t - t_0)}$$

If $y_0 > 0$, the solution is valid only

$$\frac{1}{y_0} - (t - t_0) > 0, \ \text{or } t < t_0 + \frac{1}{y_0}$$

Comparison:

	linear	nonlinear
Problem	$y' + p(t)y = g(t)$ $y(t_0) = y_0$	$y' = f(t, y)$ $y(t_0) = y_0$
General Soln formula	$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right]$ $\mu = e^{\int p(t)dt}$	non-available
Existence and Uniqueness	Theorem 1	Theorem 2
Interval of definition	global: determine by $p(t)$ and $g(t)$	local, not indicated in $f(t, y)$
Examples	$y' + \frac{2}{t}y = 4t$ $y = t^2 + \frac{C}{t^2}$	$y' = y^2$ $y = -\frac{1}{t+C}$

Review of Chapters 1 and 2

Chapter 1: Introduction

- basic concepts: DE, order, linearity, homogeneity, solution, the general solution, integral curve.
- Direction field, method of isoclines.

Chapter 2: 1st order DEs $\frac{dy}{dt} = f(t, y)$

- $y' = f(t):\ y = \int f(t)dt.$
- Linear equation $y' + p(t)y = g(t)$: Integrating factor, variation of parameters.
- separate equation $y' \equiv \frac{dy}{dt} = -\frac{M(t)}{N(t)}$.
- Exact equation $M(t, y)dt + N(t, y)dy = 0$, find $\psi(t, y)$ such that

$$d\psi = M(t, y)dt + N(t, y)dy$$

- Qualitative analysis for autonomous equation $y' = f(y)$: equilibrium solution $f(y) = 0$, stability of equilibrium solutions.
- Theory (existence and uniqueness)
- Application: Water tank problems, loan problems, population dynamics (logistic model).

Examples:

1. $\dfrac{dy}{dt} = \dfrac{2t + y}{3 + 3y^2 - t}$

2. $y' = \dfrac{t^2 - 1}{y^2 + 1}$

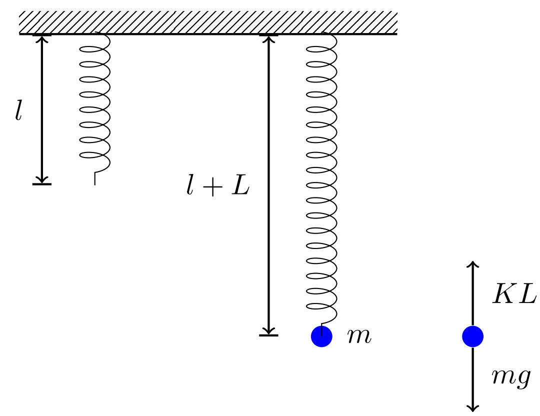
3. $\dfrac{dy}{dt} = y(1 - y)$

Chapter 3

Second Order Linear Equations

Two important areas of application of 2nd order linear equations are in the fields of mechanical and electrical oscillations.

Example Mechanical system of spring.



gravitational force: $F_g = mg$

m: mass of the ball, g acceleration due to gravity.

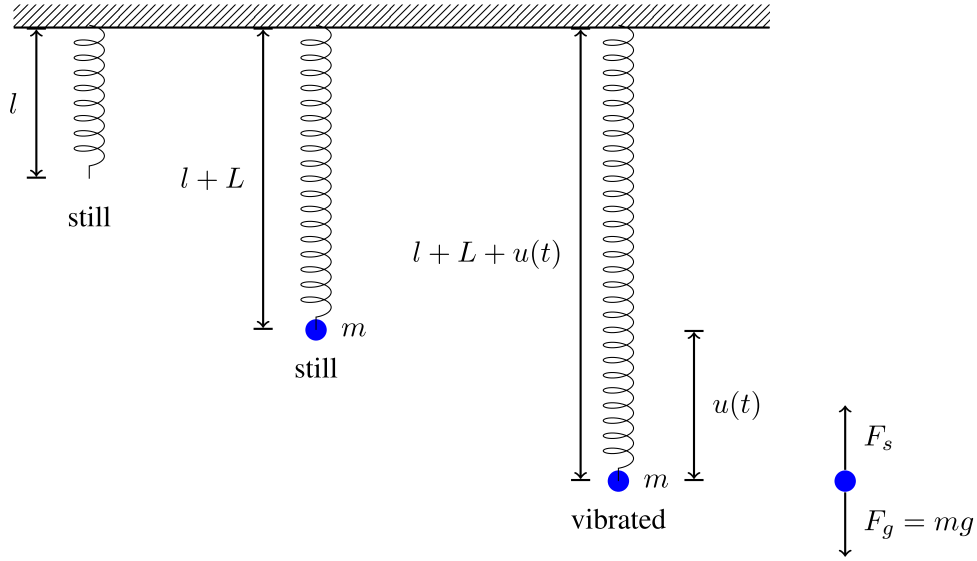
Spring force: $F_s = -kL$

L: elongation. k: constant of spring.

Hooke's Law: Spring force is proportional to the displacement of the spring (L). The direction of the force is opposite to the direction of the motion.

when the ball is in equilibrium,

$$mg = kL, \Rightarrow \; k = \frac{mg}{L}.$$



<u>**Newton's Law of motion**</u> (neglect damping or resistive force such as air resistance)

$$\boxed{mu''(t) = f(t)}$$

$F_g = mg$, $F_s = -k(L + u(t))$, $f(t) = F_g + F_s = mg - k(L + u)$.

$mu''(t) = mg - k(L + u(t))$

$mu''(t) = mg - \frac{mg}{L}(L + u(t))$

$$\begin{cases} u''(t) = -\frac{g}{L}u(t) \\ u(0) = 0.1L \\ u'(0) = 0 \end{cases}$$

Say: $L = 1$, $g = 9.8$

$$\begin{cases} u''(t) = -9.8u(t) \\ u(0) = 0.1 \\ u'(0) = 0 \end{cases}$$

Generally speaking, 2nd order DE:

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$$

2nd order linear DE:

$$\begin{cases} y'' + p(x)y' + q(x)y = g(x) \\ y(x_0) = y_0 \\ y'(x_0) = y_0' \end{cases}$$

3.1 Second Order linear homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0 \quad (a \neq 0)$$

Example 1: $y'' = y$

Possible solutions: e^x, e^{-x}, C_1e^x, C_2e^{-x};

General solution $y = C_1e^x + C_2e^{-x}$.

For $ay'' + by' + cy = 0$, seek for solution in form $y = e^{rx}$, r to be determined.

$(ar^2 + br + c)e^{rx} = 0$

$ar^2 + br + c = 0 \quad (characteristicequation)$

$r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac > 0$, then r_1, r_2 real and different, then e^{r_1x} and e^{r_2x} are two different solutions.

$$\boxed{y = C_1e^{r_1x} + C_2e^{r_2x}}$$

C_1 and C_2 can be determined by $y(x_0) = y_0$, $y'(x_0) = y'_0$.

Example 1 $y'' + 4y' + 3y = 0$, $y(0) = 2$, $y'(0) = -1$

$r^2 + 4r + 3 = 0$

$(r + 3)(r + 1) = 0$

$r = -3, \quad r = -1$

$\Rightarrow y(x) = C_1e^{-3x} + C_2e^{-x}$

$$\begin{cases} y(0) = C_1e^0 + C_2e^0 = C_1 + C_2 = 2 \\ y'(x) = -3C_1e^{-3x} - C_2e^{-x} \\ y'(0) = -3C_1 - C_2 = -1 \\ \Rightarrow C_1 = -\dfrac{1}{2}, \quad C_2 = \dfrac{5}{2} \\ y(x) = -\dfrac{1}{2}e^{-3x} + \dfrac{5}{2}e^{-x} \end{cases}$$

Example 2 Find the general solution of $2y'' - 3y' + y = 0$

$2r^2 - 3r + 1 = 0$

$(r - 2)(r - 1) = 0$

$r = 2 \quad r = 1$

$\Rightarrow y(x) = C_1e^{2x} + C_2e^x$

Example 3: $y'' = 9.8y$, $y(0) = 0.1$, $y''(0) = 0$

$r^2 - 9.8r = 0$
$r(r - 9.8) = 0$
$r = 0, \quad r = 9.8$
$y(x) = C_1 e^{0x} + C_2 e^{9.8x} = C_1 + C_2 e^{9.8x}$

$y(0) = C_1 + C_2 = 0.1$, $y''(x) = 9.8^2 C_2 e^{9.8x}$, $y''(0) = 9.8^2 C_2 = 0.1$

3.2 Solutions of Linear Homogeneous Equations, the Wronskian

Recall $\begin{cases} ay'' + by' + cy = 0 \\ ar^2 + br + c = 0 \end{cases}$ r_1, r_2 real, different. Then $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

<u>**Linear homogeneous equation**</u>:

$$\begin{cases} y'' + p(x)y' + q(x)y = 0 \\ y(x_0) = y_0 \\ y'(x_0) = y'_0 \end{cases}$$

where $p(x)$, $q(x)$ are defined in an interval $I = (\alpha, \beta)$ and $x_0 \in (\alpha, \beta)$.

<u>**Theorem 2.1**</u> (Existence and Uniqueness).

If $p(x)$ and $q(x)$ are continuous in I, then there is exactly one solution $y = \phi(x)$ of the IVP (Initial Value Problem), and the solution exists throughout the interval I.

<u>**Theorem 2.2**</u> (Superposition Principle)

If $y_1(x)$ and $y_2(x)$ are two solutions of

$$y'' + p(x)y' + q(x)y = 0$$

Then the linear combination $C_1 y_1(x) + C_2 y_2(x)$ is also a solution for any values of the constants C_1 and C_2.

Proof:

General Solution: by Krammer's rule

$y(x) = C_1 y_1(x) + C_2 y_2(x)$
$C_1 y_1(x_0) + C_2 y_2(x_0) = y_0$
$C_1 y_1'(x_0) + C_2 y_2'(x_0) = y_0'$

$$C_1 = \frac{\begin{vmatrix} y_0 & y_2(x_0) \\ y_0' & y_2'(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}}, \quad C_2 = \frac{\begin{vmatrix} y_1(x_0) & y_0 \\ y_1'(x_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}}$$

$$\boxed{w = w[y_1, y_2](x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}}$$ Wronskian determinant of y_1 and y_2

Theorem 2.3 (General solution)

If y_1 and y_2 are two solution of

$$y'' + p(x)y' + q(x)y = 0$$

and if there is a point where the wronskian of y_1 and y_2 is nonzero, then the general solution of the DE is

$$\boxed{y = C_1 y_1(x) + C_2 y_2(x)}$$

where C_1 and C_2 are arbitrary constants. y_1 and y_2 are said to form a fundamental set of solutions.

Example $y'' + 4y' + 3y = 0$.

$r^2 + 4r + 3 = 0$
$(r + 3)(r + 1) = 0$
$r = -3, \quad r = -1$

Two particular solutions $y_1(x) = e^{-3x}$ and $y_2(x) = e^{-x}$. At $x_0 = 0$, $y_1(x_0) = e^0 = 1$ and $y_2(x_0) = e^0 = 1$. So

$$y = C_1 e^{-3x} + C_2 + e^{-x}$$

is the fundamental solution.

Def: Linear Independence of functions

Two functions f and g are said to be linearly dependent on interval I if there exist two constants k_1 and k_2, not both zero, such that

$$k_1 f(t) + k_2 g(t) = 0, \quad \forall t \in I$$

f and g are said to be linearly independent on I if they are not linearly dependent.

Example: $f(t) = 3t$, $g(t) = |t|$, (a) $I = (0, \infty)$, (b) $I = (-\infty, \infty)$.

$k_1 \cdot 3t + k_2|t| = 0, \quad \forall t \in I$

(a)$I = (0, \infty)$: $k_1 \cdot 3t + k_2 t = 0 \Rightarrow (3k_1 + k_2)t = 0 \Rightarrow k_2 = -3k_1$. So as long as $k_2 = -3k_1$, for example $k_2 = -3$ and $k_1 = 1$, then $k_1 \cdot 3t + k_2 t = 0, \quad \forall t \in I$, which implies $f(t)$ and $g(t)$ are *linear dependent* on $(0, \infty)$.

(b)$I = (-\infty, \infty)$: $\begin{cases} k_1 \cdot 3t + k_2 t = 0, & t \in [0, \infty) \\ k_1 \cdot 3t - k_2 t = 0, & t \in (-\infty, 0) \end{cases}$

$\Rightarrow \;\; k_2 = -3k_1$ and $k_2 = 3k_1$, $\Rightarrow k_1 = k_2 = 0$, which implies $f(t)$ and $g(t)$ are *linear independent* on $(-\infty, \infty)$.

Linear independence – The Wronkskian

Theorem 3.1 If y_1 and y_2 are two solutions of

$$y'' + p(x)y' + q(x)y = 0$$

where p and q are continuous on an open interval I, then the Wronskian $W[y_1, y_2](x)$ is given by

$$\boxed{W[y_1, y_2](x) = Ce^{-\int p(x)dx}}$$

where C is a certain constant that depends on y_1 and y_2, but not on x. Furthermore, $w[y_1, y_2](x)$ is either zero for all $x \in I$ (if $C = 0$) or else is never zero in I (if $c \neq 0$)

Proof.

$(y_1'' + p(x)y_1' + q(x)y_1) \cdot y_2 = 0$
$(y_1'' + p(x)y_2' + q(x)y_2) \cdot y_1 = 0$
$w = y_1(x)y_2'(x) - y_2(x)y_1'(x)$

■

Example Find the Wronskian of $x^2 y'' - (x+2)y' + (x+2)y = 0$ without solving this DE.

First, we need to divide the differential equation by the first coefficient of the second derivative term:x^2. This gives us $y'' - \dfrac{x+2}{x^2}y' + \dfrac{x+2}{x^2} = 0$. It turns out $p(x) = \dfrac{x+2}{x^2}$.

$\displaystyle\int \frac{x+2}{x^2}dx = \int \frac{x}{x^2} + \frac{2}{x^2}dx = \ln|x| - \frac{2}{x}$

$W[y_1, y_2](x) = Ce^{-\int p(x)dx} = Ce^{\ln|x| - \frac{2}{x}} = C|x|e^{-\frac{2}{x}}$

Euler's Formula

Motivation: e^{3+6i}

Taylor series of e^x

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} \\
&= \boxed{\cos x + i \sin x}
\end{aligned}
$$

Example 1: $e^{\lambda + i\mu x} = e^{\lambda}(\cos(\mu x) + i \sin(\mu x))$

$e^{3+6i} = e^3(\cos(6) + i\sin(6))$

Property: $\dfrac{de^{rx}}{dx} = re^{rx}$

3.3 Complex Roots of the characteristic Equations

Consider $ay'' + by' + cy = 0$. $y = e^{rx}$, r to be determined.

Characteristic equation: $ar^2 + br + c = 0$.

If $b^2 - 4ac < 0$, then

$$
r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu
$$

$$
\lambda = \frac{-b}{2a}, \quad \mu = \frac{\sqrt{|b^2 - 4ac|}}{2a}
$$

$$
\boxed{y_1(x) = e^{(\lambda + i\mu)x} = e^{\lambda x}(\cos \mu x + i \sin \mu x)}
$$

$$
\boxed{y_2(x) = e^{(\lambda - i\mu)x} = e^{\lambda x}(\cos \mu x - i \sin \mu x)}
$$

$$
\begin{aligned}
W[y_1, y_2](x) &= \begin{vmatrix} e^{(\lambda+i\mu)x} & e^{(\lambda-i\mu)x} \\ (\lambda+i\mu)e^{(\lambda+i\mu)x} & (\lambda-i\mu)e^{(\lambda-i\mu)x} \end{vmatrix} \\
&= (\lambda - i\mu)e^{2\lambda x} - (\lambda + i\mu)e^{2\lambda x} \\
&= -2i\mu e^{2\lambda x} \neq 0
\end{aligned}
$$

$y(x) = C_1 y_1(x) + C_2 y_2(x)$

$$\begin{aligned} y(x) &= C_1 e^{\lambda x}(\cos \mu x + i \sin \mu x) + C_2 e^{\lambda x}(\cos \mu x - i \sin \mu x) \\ &= (C_1 + C_2) e^{\lambda x} \cos \mu x + i(C_1 - C_2) e^{\lambda x} \sin \mu x \end{aligned}$$

Define: $\tilde{C_1} = C_1 + C_2$, $\tilde{C_2} = i(C_1 - C_2)$.

$$\boxed{y(x) = \tilde{C_1} e^{\lambda x} \cos \mu x + \tilde{C_2} \sin \mu x}$$

$\tilde{y_1}(x) = e^{\lambda x} \cos \mu x$

$\tilde{y_2}(x) = e^{\lambda x} \sin \mu x$

Here $\tilde{y_1}$ and $\tilde{y_2}$ are two real solutions.

Example 1 $y'' = -y$

$y'' + y = 0$

$r^2 + 1 = 0$

$b^2 - 4ac = -4 < 0$

$\lambda = -\dfrac{b}{2a} = 0, \quad \mu = \dfrac{\sqrt{|b^2 - 4ac|}}{2a} = 1$

$y_1(x) = e^{ix} = \cos x + i \sin x, \quad y_2(x) = e^{-ix} = \cos x - i \sin x$

$y(x) = C_1 e^{ix} + C_2 e^{-ix}$

$\tilde{C_1} = C_1 + C_2, \quad \tilde{C_2} = i(C_1 - C_2)$

$\tilde{y_1}(x) = \cos x, \quad \tilde{y_2} = \sin x.$

Example 2 $y'' + y' + 1.25y = 0$, $y(0) = 3$, $y'(0) = 1$

$r^2 + r + 1.25 = 0$

$b^2 - 4ac = 1 - 5 = -4$

$\lambda = -\dfrac{b}{2a} = -\dfrac{1}{2}, \quad \mu = \dfrac{\sqrt{|4|}}{2} = 1$

$y_1 = e^{-\frac{1}{2}x + ix} = e^{-\frac{1}{2}x}(\cos x + i \sin x), \quad y_2 = e^{-\frac{1}{2}x - ix} = e^{-\frac{1}{2}x}(\cos x - i \sin x)$

$y(x) = C_1 y_1(x) + C_2 y_2(x)$

$\tilde{C_1} = C_1 + C_2 \quad \tilde{C_2} = i(C_1 - C_2)$

$\tilde{y_1}(x) = e^{-\frac{1}{2}x} \cos x, \quad \tilde{y_2}(x) = e^{-\frac{1}{2}x} \sin x.$

3.4 Repeated Roots

Consider:

$ay'' + by' + cy = 0$
$ar^2 + br + c = 0$
$r_{1,2} = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

- $b^2 - 4ac > 0$: r_1 and r_2 real and different.
- $b^2 - 4ac < 0$: r_1 and r_1 complex and different.
- $b^2 - 4ac = 0$: $r_1 = r_2 = -\dfrac{b}{2a}$ real.

In the last case, $y_1(x) = e^{-r_1 x} = y_2 = e^{-r_2 x}$.

$W[y_1, y_2](x) = 0$

General solution?

To find the general solution, we have to find $y_2(x)$ which is different from $y_1(x)$.

Assume that $y_2(x)$ has the form

$$\upsilon(x) y_1(x) = \upsilon(x) e^{r_1 x} = \upsilon(x) e^{-\frac{b}{2a}x}$$

where $\upsilon(x)$ is to be determined.

$y_2' = \upsilon' e^{-\frac{b}{2a}x} - \dfrac{b}{2a} \upsilon e^{-\frac{b}{2a}x}$
$y_2'' = \upsilon'' e^{-\frac{b}{2a}x} - \dfrac{b}{a} \upsilon' e^{-\frac{b}{2a}x} + \dfrac{b^2}{4a^2} \upsilon e^{-\frac{b}{2a}x}$
$a[\upsilon'' e^{-\frac{b}{2a}x} - \dfrac{b}{a} \upsilon' e^{-\frac{b}{2a}x} + \dfrac{b^2}{4a^2} \upsilon e^{-\frac{b}{2a}x}] + b[\upsilon' e^{-\frac{b}{2a}x} - \dfrac{b}{2a} \upsilon e^{-\frac{b}{2a}x}] + c\upsilon e^{-\frac{b}{2a}x} = 0$
$a\upsilon'' + \left(\underbrace{c - \dfrac{b^2}{4a}}_{=0} \right) \upsilon = 0$
$\upsilon'' = 0$
$\upsilon(x) = C_1 x + C_2$
$y_2(x) = C_1 x e^{-\frac{b}{2a}x} + C_2 e^{-\frac{b}{2a}x}$

$$\begin{cases} y_1(x) = e^{-\frac{b}{2a}x} \\ y_2(x) = xe^{-\frac{b}{2a}x} \end{cases}$$

General solution: $y(x) = \boxed{C_1 e^{-\frac{b}{2a}x} + C_2 x e^{-\frac{b}{2a}x}}$

Summary General solution of $ay'' + by' + cy = 0$

$ar^2 + br + c = 0$

$r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- $b^2 - 4ac > 0$: $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
- $b^2 - 4ac < 0$: $\lambda = -\dfrac{b}{2a}, \quad \mu = \dfrac{\sqrt{|b^2 - 4ac|}}{2a}$
  $y(x) = e^{\lambda x}(C_1 \cos \mu x + C_2 \sin \mu x)$
- $b^2 - 4ac = 0$: $y(x) = C_1 e^{-\frac{b}{2a}x} + C_2 x e^{-\frac{b}{2a}x}$

Examples:

- $y'' - 2y' + 10y = 0$
- $y'' = 4y$
- $9y'' + 6y' + y = 0$
- $y'' + 4y' + 2y = 0$
- $y'' = -4y$

3.5 Nonhomogeneous Equation, Method of undetermined coefficients

Homogeneous DE: $ay'' + by' + cy = 0$

Nonhomogeneous DE: $ay'' + by' + cy = g(x)$?

Theorem: The general solution of nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x)$$

can be written in the form

$$y = C_1 y_1(x) + C_2 y_2(x) + Y(x)$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

C_1 and C_2 are arbitrary constants, and Y is some specific solution of the nonhomogeneous equation.

Procedure:

- Find y_1 and y_2
- Find $Y(x)$
- $y = C_1y_1(x) + C_2y_2(x) + Y(x)$

How to find $Y(x)$:

The Method of undetermined coefficents

$ay'' + by' + cy = g_1(x) + g_2(x)$

Splitting $\Rightarrow$

If $Y_1(x)$ is the solution of $ay'' + by' + cy = g_1(x)$,

$Y_2(x)$ is the solution of $ay'' + by' + cy = g_2(x)$

then $Y_1(x) + Y_2(x)$ is a solution of $ay'' + by' + cy = g_1(x) + g_2(x)$.

Examples

$y'' - 2y' - 3y = 3e^{2x}$

$y'' + 4y = 3\sin 2x$, $y(0) = 2$, $y'(0) = -1$

$y'' - 3y' - 4y = -8e^x \cos 2x$

3.6 Variation of Parameters

Find a specific solution of the non-homogeneous equation.

Example Find the general solution of $y'' + 4y = \sin x$

Solution Find y_1 and y_2 of $y'' + 4y = 0$, $\Rightarrow$

$y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$, $y(x) = C_1y_1(x) + C_2y_2(x)$(*)

The basic idea in the method of variation of parameters is to replace the constants C_1 and C_2 in (*) by functions $u_1(x)$ and $u_2(x)$, resp., and then to determine these functions so that

$$Y(x) = u_1(x)\cos 2x + u_2(x)\sin 2x$$

is a particular solution of the non-homogeneous equation.

$$\begin{aligned} Y' &= u_1'\cos 2x + u_2'\sin 2x - 2u_1\sin 2x + 2u_2\cos 2x \\ Y'' &= u_1''\cos 2x + u_2''\sin 2x - 2u_1'\sin 2x + 2u_2'\cos 2x - 2u_1'\sin 2x \\ &\quad + 2u_2'\cos 2x - 4u_1\cos 2x - 4u_2\sin 2x \\ &= u_1''\cos 2x + u_2''\sin 2x - 4u_1'\sin 2x + 4u_2'\cos 2x - 4u_1\cos 2x - 4u_2\sin 2x \end{aligned}$$

Since $Y'' + 4Y = \sin x \Rightarrow$

$$\begin{aligned} &[u_1''\cos 2x + u_2''\sin 2x - 4u_1'\sin 2x + 4u_2'\cos 2x - 4u_1\cos 2x - 4u_2\sin 2x] \\ &\quad + 4[u_1\cos 2x + u_2\sin 2x] = \sin x \\ &u_1''\cos 2x + u_2''\sin 2x - 4u_1'\sin 2x + 4u_2'\cos 2x = \sin 2x \end{aligned}$$

which is one equation for unknown functions u_1 and u_2. To determine them, we need to seek for another equation or relation.

Let us impose one more restriction $u_1'\cos 2x + u_2'\sin 2x = 0$, then

$$Y' = -2u_1\sin 2x + 2u_2\cos 2x$$

$$Y'' = -2u_1'\sin 2x + 2u_2'\cos 2x - 4u_1\cos 2x - 4u_2\sin 2x$$

Since $Y'' + 4Y = \sin x \Rightarrow$

$$-2u_1'\sin 2x + 2u_2'\cos 2x = \sin x$$

$$\begin{aligned} &u_1' = -\frac{\sin 2x}{\cos 2x}u_2' \\ &2u_2'\frac{(\sin 2x)^2}{\cos 2x} + 2u_2'\cos 2x = \sin x \\ &u_2' = \frac{1}{2}\sin x\cos 2x \end{aligned}$$

$$\begin{aligned} u_2 &= \frac{1}{2}\int \sin x\cos 2x dx \\ &= \frac{1}{2}\int \frac{1}{2}[\sin(x+2x) - \sin(x-2x)]dx \\ &= -\frac{1}{12}\cos 3x - \frac{1}{4}\cos x + C_3 \end{aligned}$$

$$u_1' = -\frac{1}{2}\sin x \sin 2x$$
$$u_1 = -\frac{1}{2}\int \sin x \sin 2x dx$$
$$= -\frac{1}{4}\sin x + \frac{1}{12}\sin 3x + C_4$$

$$Y(x) = [-\frac{1}{4}\sin x + \frac{1}{12}\sin 3x]\cos 2x + [-\frac{1}{12}\cos 3x - \frac{1}{4}\cos x]\sin 2x$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + Y(x)$$

Summary To solve $y'' + p(x)y' + q(x)y = g(x)$ (*)

Suppose $y_1(x)$ and $y_2(x)$ are solutions of $y'' + p(x)y' + q(x)y = 0$

$Y(x)$ is a particular solution of (*), in the form of $Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$. Take the first derivative of Y, we have

$$Y' = u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2'.$$

Suppose $u_1'y_1 + u_2'y_2 = 0$, then $Y' = u_1y_1' + u_2y_2'$.

$$Y'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$$

Since $Y'' + pY' + qY = g$

$$u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' + p[u_1y_1' + u_2y_2'] + [u_1y_1 + u_2y_2] = g$$

$$\Rightarrow u_1'y_1' + u_2'y_2' + u_1 \underbrace{[y_1'' + py_1' + qy_1]}_{=0} + u_2 \underbrace{[y_2'' + py_2 + qy_2]}_{=0} = g$$

$$\Rightarrow \boxed{u_1'y_1' + u_2'y_2' = g}$$

By $\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g \end{cases}$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2 g}{W[y_1, y_2](x)}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 g}{W[y_1, y_2](x)}$$

$$u_1 = -\int \frac{gy_2}{W[y_1, y_2](x)}dx,\ u_2 = \int \frac{gy_1}{W[y_1, y_2](x)}dx$$

$$Y(x) = -y_1 \int \frac{gy_2}{W[y_1, y_2](x)} dx + y_2 \int \frac{gy_1}{W[y_1, y_2](x)} dx$$

$$\boxed{y = C_1 y_1(x) + C_2 y_2(x) - y_1 \int \frac{gy_2}{W[y_1, y_2](x)} dx + y_2 \int \frac{gy_1}{W[y_1, y_2](x)} dx}$$

3.7 Variation of Parameters

Theorem If y_1 and y_2 are a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

then a particular solution of

$$y'' + p(x)y' + q(x)y = g(x)$$

is

$$Y(x) = -y_1 \int \frac{gy_2}{W[y_1, y_2](x)} dx + y_2 \int \frac{gy_1}{W[y_1, y_2](x)} dx$$

and general solution is

$$y = C_1 y_1(x) + C_2 y_2(x) + Y(x)$$

Example 1 $y'' + 4y = \sin x$

$y'' + 4y = 0$ gives us $y_1 = \cos 2x$, $y_2 = \sin 2x$

$$W[y_1, y_2](x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$\int \frac{gy_2}{W[y_1, y_2](x)} dx = \int \frac{\sin 2x \sin x}{2} dx = \frac{1}{4}\sin x - \frac{1}{12}\sin 3x + C_1$$

$$\int \frac{gy_1}{W[y_1, y_2](x)} dx = \int \frac{\cos 2x \sin x}{2} dx = -\frac{1}{4}\cos x - \frac{1}{12}\cos 3x + C_2$$

A particular solution is

$$Y(x) = -\cos 2x[\frac{1}{4}\sin x - \frac{1}{12}\sin 3x] + \sin 2x[-\frac{1}{4}\cos x - \frac{1}{12}\cos 3x]$$

General solution is

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + Y(x)$$

Example 2: $y'' - y' - 2y = 2e^{-x}$

To find a set fundamental solutions of $y'' - y' - 2y = 0$,

$r^2 - r - 2 = 0$
$(r-2)(r+1) = 0$
$r = 2, \quad r = -1$
$y_1 = e^{2x}, \quad y_2 = e^{-x}$

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix} = -e^x - 2e^x = -3e^x$$

$$\int \frac{gy_2}{W[y_1, y_2](x)} dx = \int \frac{2e^{-x} \cdot e^{-x}}{-3e^x} dx = -\frac{2}{3} \int e^{-3x} dx = \frac{2}{9} e^{-3x}$$

$$\int \frac{gy_1}{W[y_1, y_2](x)} dx = \int \frac{2e^{-x} \cdot e^{2x}}{-3e^x} dx = \int -\frac{2}{3} dx = -\frac{2x}{3}$$

A particular solution is $Y = -e^{2x} \cdot \frac{2}{9} e^{-3x} - e^{-x} \cdot \frac{2x}{3} = -\left(\frac{2}{9} + \frac{2x}{3}\right) e^{-x}$

General solution is $y(x) = C_1 e^{2x} + C_2 e^{-x} + Y(x)$

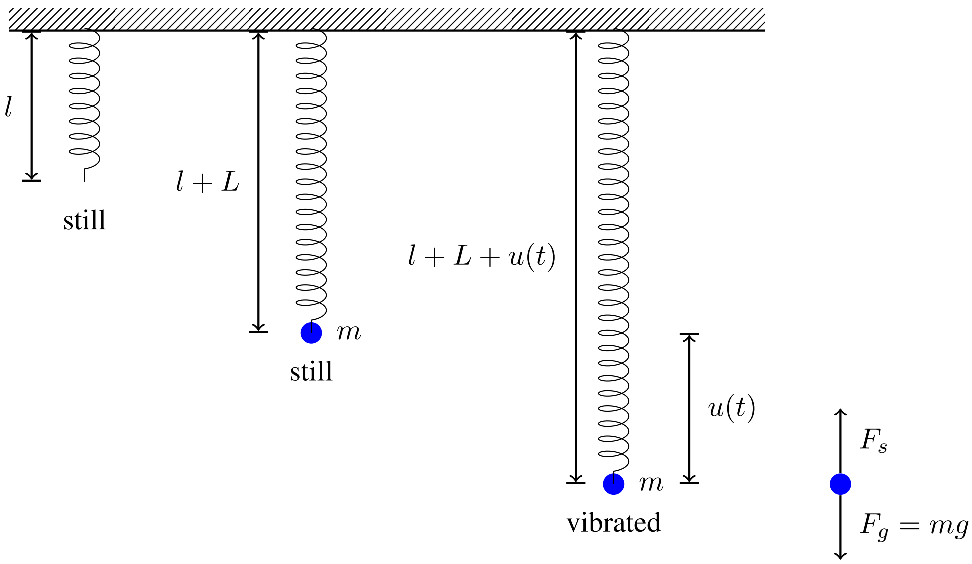
Example 3: $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ (heat equation)

$u = e^{-n^2 t} y(x), n = 0, 1, 2, ...$

$$-n^2 e^{-n^2 t} y(x) = e^{-n^2 t} \frac{d^2 y}{dx^2}$$

$$\frac{d^2 y}{dx^2} + n^2 y(x) = 0$$

3.8 Mechanical and Electrical Vibrations



undamped free vibration

m: mass. k: spring constant.

$$\boxed{mu'' + ku = 0}$$

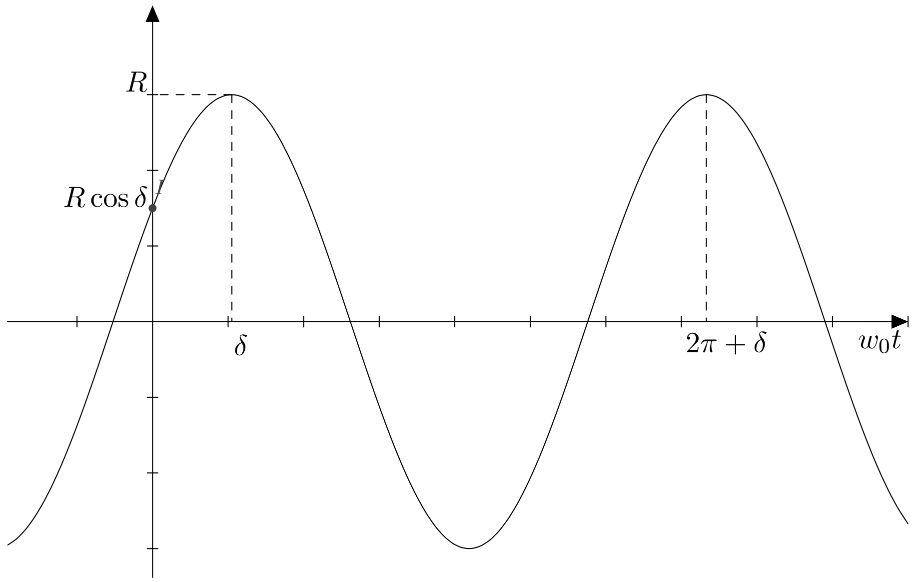
$$\begin{aligned} u &= A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t \\ &= A\cos w_0 t + B\sin w_0 t \quad w_0 = \sqrt{\frac{k}{m}} \\ &= R\cos(w_0 t - \delta) \end{aligned}$$

Here $A = R\cos\delta$, $B = R\sin\delta$ or $R = \sqrt{A^2 + B^2}$, $\tan\delta = \dfrac{B}{A}$.

$w_0 = \sqrt{\dfrac{k}{m}}$: in radians per unit time, natural frequency of a vibration.

$T = \dfrac{2\pi}{w_0}$: period of a vibration.

R: amplitude.



δ: phase or phase angle.

<u>Damped free vibration</u>

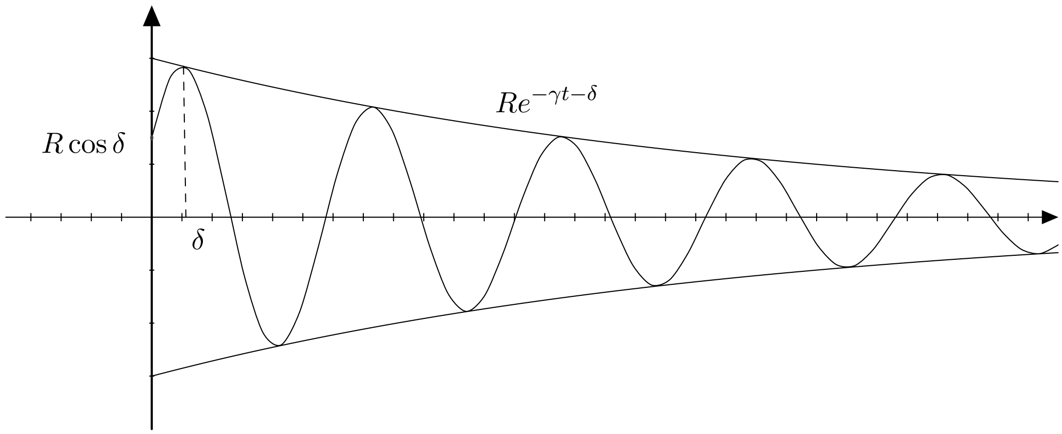
$$mu'' + \gamma u' + ku = 0, \quad \gamma > 0$$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

$\gamma^2 - 4km > 0$: $u = Ae^{r_1 t} + Be^{r_2 t}$

$\gamma^2 - 4km = 0$: $u = (A + Bt)e^{-\frac{\gamma t}{2m}}$

$\gamma^2 - 4km < 0$: $u = e^{-\frac{\gamma}{2m}t}(A\cos\mu t + B\sin\mu t) = Re^{-\frac{\gamma}{2m}t}\cos(\mu t + \delta)$, $\mu = \dfrac{\sqrt{4km - \gamma^2}}{2m} > 0$, $R = \sqrt{A^2 + B^2}$, $\tan\delta = \dfrac{A}{B}$. μ: quasi frequency, $T_d = \dfrac{2\pi}{\mu}$: quasi-period.



Damped vibration with external force

$$mu'' + \underbrace{\gamma}_{\text{damped constant}} u' + ku = \underbrace{F_0 \cos wt}_{\text{external periodical force}}$$

(a) $\gamma = 0$, $w \neq w_0$

$$\begin{aligned} u &= C_1 \cos w_0 t + C_2 \sin w_0 t + \frac{F_0}{m(w_0^2 - w^2)} \cos wt \\ &= R\cos(w_0 t - \delta) + \frac{F_0}{m(w_0^2 - w^2)} \cos wt \end{aligned}$$

$u(0) = 0$, $u'(0) = 0$ (no initial displacement, no initial velocity)

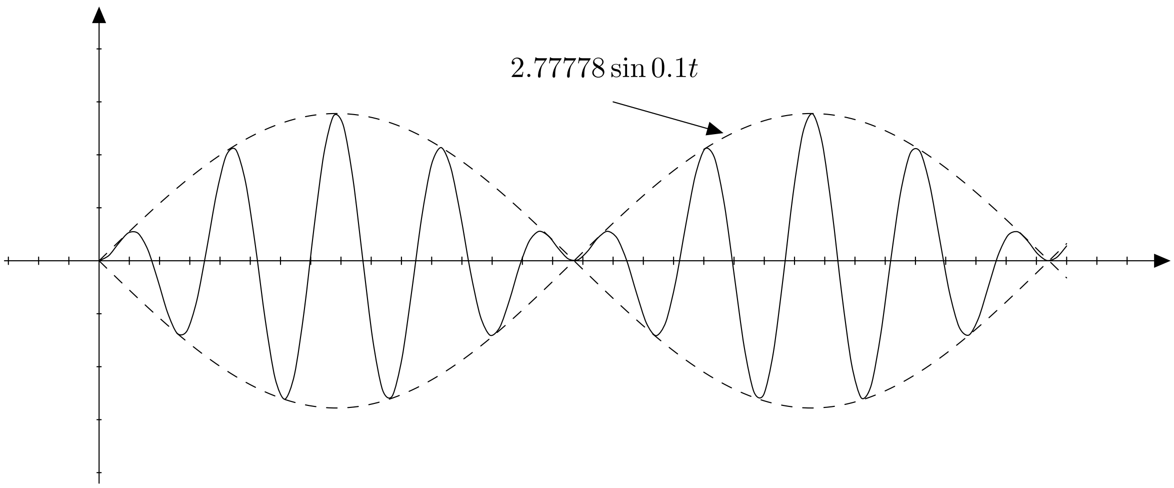
$$\begin{cases} R\cos\delta + \dfrac{F_0}{m(w_0^2 - w^2)} = 0 \\ -Rw_0 \sin\delta = 0 \end{cases} \quad \Rightarrow \begin{cases} R = -\dfrac{F_0}{m(w_0^2 - w^2)} \\ \delta = 0 \end{cases}$$

$$\begin{aligned} u &= \frac{F_0}{m(w_0^2 - w^2)}(\cos wt - \cos w_0 t) \\ &= \left(\frac{2F_0}{m(w_0^2 - w^2)} \sin \frac{(w_0 - w)t}{2} \right) \sin \frac{w_0 + w}{2} t \end{aligned}$$

Example 1

$u'' + u = 0.5 \cos 0.8t$
$u(0) = 0, \quad u'(0) = 0$
$u = 2.77778 \sin 0.1t \sin 0.9t$

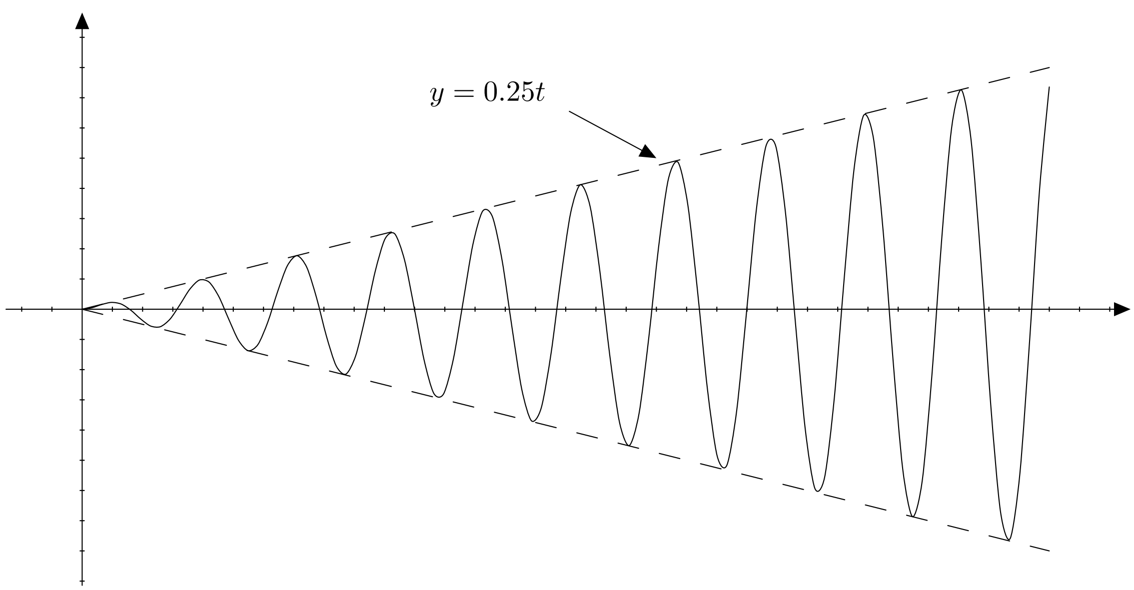
Graph of $u = 2.77778 \sin 0.1t \sin 0.9t$



(b) $w = w_0$ $u = C_1 \cos w_0 t + C_2 \sin w_0 t + \dfrac{F_0}{2mw_0} t \sin w_0 t$

$t \to \infty \quad u \to \infty$.

Example 2 $u'' + u = 0.5 \cos t$, $u(0) = 0$, $u'(0) = 0$ then $u = 0.25t \sin t$.



This phenomenon is called "resonance".

Review of Chapter 8 and 3

Chapter 8

$$\begin{cases} \dfrac{du}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Given $h > 0$

Euler's method:

$y_{n+1} = y_n + hf(t_n, y_n)$
$F_{n+1} = t_n + h$
$y_{n+1} \simeq \phi(t_{n+1})$
$|y_{n+1} - \phi(t_{n+1})| \leq Ch$

Heun's method:

$y_{n+1} = y_n + \dfrac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n) + hf(t_n, y_n)]$
$t_{n+1} = t_n + h$

Chapter 3:

$$\begin{cases} ay'' + by' + cy = g(x) \\ y(x_0) = y_0, \quad y'(x_0) = y_0' \end{cases}$$

I. The Wronskian

definition

$$W[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

if $W[y_1, y_2](x) \neq 0$, then we say y_1 and y_2 are linearly independent.

Theorem Let y_1 and y_2 be two solutions of

$$y'' + p(x)y' + q(x)y = 0$$

Then $W[y_1, y_2](x) = ce^{-\int p(x)dx}$

II. Homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0$$

characteristic equation:

$$ar^2 + br + c = 0$$
$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. r_1 and r_2 are real and different, general solutions is

$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ (exponential)

2. $r_1 = r_2$ real

$$y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x}$$

3. r_1 and r_2 different complex

$$r_{1,2} = \lambda \pm \mu i, \quad \lambda = -\frac{b}{2a}, \quad \mu = \frac{\sqrt{|b^2 - 4ac|}}{2a}$$

$y(x) = C_1 e^{\lambda x} \cos \mu x + C_2 e^{\lambda x} \sin \mu x$, C_1: real C_2: complex

III. Nonhomogeneous Equations: method of undetermined coefficients

$$y'' + p(x)y' + q(x)y = g(x)$$
$$Y(x) = -y_1(x) \int \frac{y_2(x)g(x)}{W[y_1, y_2](x)} dx + y_2(x) \int \frac{y_1(x)}{W[y_1, y_2](x)} dx$$
$$y(x) = C_1 y_1(x) + C_2 y_2(x) + Y(x)$$

IV (1) The Euler's Formula

$$e^{i\mu x} = \cos \mu x + i \sin \mu x$$
$$\cos \mu x = \frac{1}{2}[e^{i\mu x} + e^{-i\mu x}]$$
$$\sin \mu x = -\frac{i}{2}[e^{i\mu x} - e^{-i\mu x}]$$

(2) Existence Theorem.

V: Application: Mechanical Vibrations

1. undamped free
2. damped free
3. undamped force
    - beats

- resonance

Examples

1. $y' = t^2 + y^2$, $y(0) = 1$. Solutions at $t = 0.1, 0.2$ with Euler method with $h = 0.05$.

2. $y'' - 2y' + y = x$, $y(0) = 0$, $y'(0) = 1$.
   $y'' - 2y' + y = x \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$, $r_1 = r_2 = 1$.
   $y_1(x) = e^x$, $y_2(x) = xe^x$
   $$W[y_1, y_2](x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$
   $$\int \frac{y_2(x)g(x)}{W[y_1, y_2](x)} dx = \int \frac{x^2 e^x}{e^{2x}} dx = \int x^2 e^{-x} dx$$
   Using method of integration by parts:
   $$v = x^2, \quad du = e^{-x}$$
   $$dv = 2xdx, \quad u = -e^{-x}$$
   $$\int x^2 e^{-x} dx = -x^2 e^{-x} + \int 2xe^{-x} dx$$
   $$v = 2x, \quad du = e^{-x}$$
   $$dv = 2dx, \quad u = -e^{-x}$$
   $$\int 2xe^{-x} dx = -2xe^{-x} + 2\int e^{-x} dx = -2xe^{-x} - 2e^{-x}$$
   $$\int \frac{y_2(x)g(x)}{W[y_1, y_2](x)} dx = \int \frac{x^2 e^x}{e^{2x}} dx = \int x^2 e^{-x} dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x}$$
   $$\int \frac{y_1(x)g(x)}{W[y_1, y_2](x)} dx = \int xe^{-x} dx = -xe^{-x} - e^{-x}$$
   $Y(x) = x^2 + 2x + 2 - x^2 - x = x + 2$.
   $y = C_1 e^x + C_2 xe^x + x + 2$. $y'(x) = C_1 e^x + C_2 e^x + C_2 xe^x + 1$
   $$\begin{cases} y(0) = C_1 + 2 = 0 \\ y'(0) = C_1 + C_2 = 1 \end{cases} \Rightarrow C_1 = -2, \quad C_2 = 3.$$

3. $2y'' + 2y' + y = 0$.
   $2r^2 + 2r + 1 = 0$, $b^2 - 4ac = 4 - 8 = -4 < 0$,
   $$\lambda = -\frac{b}{2a} = -\frac{1}{2}, \ \mu = \frac{\sqrt{|b^2 - 4ac|}}{2a} = \frac{1}{2}$$
   $$y(x) = C_1 e^{-\frac{1}{2}x} \cos \frac{1}{2}x + C_2 e^{-\frac{1}{2}x} \sin \frac{1}{2}x$$

Chapter 5

System of First Order Linear Equations

5.1 Introduction and Review of matrices

Brief introduction:

Example 1: an example of system of first order linear equaiton:

$$\begin{cases} x'(t) = x(t) + 2y(t) + \sin(t) \\ y'(t) = 2x(t) + y(t) + \cos(t) \end{cases}$$

Example 2: higher order differential equations

- $y'' = y \quad y = y(t)$. Let $x(t) = y'(t)$
$$\begin{cases} x'(t) = y \\ y'(t) = x \end{cases}$$

- $y^{(3)} = y$. Let $\begin{array}{l} x_4(t) = y \\ x_3(t) = x_4'(t) = y' \\ x_2(t) = x_3'(t) = x_4''(t) = y'' \\ x_1(t) = x_2'(t) = x_3''(t) = x_4'''(t) = y''' \end{array}$

  So the system of linear equations we are to solve is
$$\begin{cases} x_1' = x_4 \\ x_2' = x_1 \\ x_3' = x_2 \\ x_4' = x_3 \end{cases}$$

General form: $n > 0$, $x_1(t)$, $x_2(t)$,...,$x_n(t)$

$$\begin{cases} x_1' = p_{11}x_1 + p_{12}x_2 + ... + p_{1n}x_n + g_1(t) \\ x_2' = p_{21}x_1 + p_{22}x_2 + ... + p_{2n}x_n + g_2(t) \\ \vdots \\ x_n' = p_{n1}x_1 + p_{n2}x_2 + ... + p_{nn}x_n + g_b(t) \end{cases}$$

Review of Matrix Theory:

(a)Definition

- $m \times n$ matrix
  $$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})_{m\times n} = (a_{ij})$$
- vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 10 \end{bmatrix}$
- Transpose of A:
  $$A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \vdots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}_{n\times m}$$
- Conjugate : $\bar{a}_{ij} =$ Real part of $a_{ij} -$ Imaginary part of a_{ij}
- Adjoint of A: $A^* = \bar{A}^T$

(b) Particular matrices

- zero matrix: $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$
- square matrix $A = (a_{ij})_{n\times n}$
- Identity matrix $I = \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{bmatrix}_{n\times n}$ (it must be a square matrix).

- diagonal matrix: $A = \text{diag}(a_{11}, a_{22}, ..., a_{nn}) = \begin{bmatrix} a_{11} & & & & \\ & a_{22} & & 0 & \\ & & \cdots & & \\ & 0 & & a_{n-1n-1} & \\ & & & & a_{nn} \end{bmatrix}$

- upper triangular matrix $\begin{bmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & \cdots & & \\ & 0 & & a_{n-1n-1} & \\ & & & & a_{nn} \end{bmatrix}$

- lower triangular matrix $\begin{bmatrix} a_{11} & & & & \\ & a_{22} & & 0 & \\ & & \cdots & & \\ & & & a_{n-1n-1} & \\ & & & & a_{nn} \end{bmatrix}$

(c) Operations and matrices

- Equality: same size and same elements.
- addition: same size $A_{m\times n} + B_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$
- multiplication by a number: $\alpha A = (\alpha a_{ij})$
- multiplication by two matrices:
  $AB = C \quad A = (a_{ij})_{m\times n}, \quad B = (b_{jk})_{n\times q}$
  $C = (c_{ik})_{m\times q}, \quad c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$
  $(ABC) = (AB)C = A(BC)$, but $AB \neq BA$
  Examples: $Ax =?$, $x^T y =?$, $xx =?$.

5.2 Systems of Linear Algebraic Equations, Linear Independence, Eigenvalues, Eigenvectors

Systems of linear Algebraic Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

Given $Ax = b$, if $det(A) \neq 0 \;\Rightarrow\; x = A^{-1}b$

Example $n = 2$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & \frac{4}{3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{4}{3} \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = -\frac{2}{3} \\ x_2 = \frac{4}{3} \end{array}$$

Linear Independence

Def A set of k vectors $x^{(1)}, ..., x^{(k)}$ is said to be linearly dependent if there exists a set of (complex) numbers $c_1, ..., c_k$. At least one of which is nonzero, such that

$$c_1 x^{(1)} + \cdots + c_k x^{(k)} = 0 \quad (*)$$

On the other hand, if the only set $c_1, ..., c_k$ for which (*) is satisfied is $c_1 = \cdots = c_k = 0$, then $x^{(1)}, \cdots, x^{(k)}$ are said to be linearly independent.

Example: $x^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x^{(k)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x^{(3)} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$

$c_1 x^{(1)} + c_2 x^{(2)} + c_3 x^{(3)} = 0$

$$\begin{cases} c_1 + 2C_3 = 0 \\ 2C_2 + 4C_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$C_1 = -2C_3, \quad C_2 = -\frac{3}{2}C_3, \quad C_1 = -2C_3$

For example $\{C_1, C_2, C_3\} = \{-4, -3, 2\}$. So $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ are linearly dependent.

Eigenvalues and Eigenvectors

Def A is a $n \times n$ matrix. If there exist a value λ and a non-zero vector x such that

$$Ax = \lambda x$$

then λ is called an eigenvalue of A and x is an eigenvector corresponding to λ.

- if x is a eigenvector corresponding to λ, then cx is a eigenvector corresponding to $c\lambda$.
- λ satisfies $\det(A - \lambda I) = 0$.
- if A is a Hermitian matrices $(a_{ij}^*) = a_(ji)$, all eigenvalues are real.

Example 1: Find eigenvalues and eigenvectors for $A = \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & \ddots & \\ & & & & 10 \end{bmatrix}$

Example 2: $A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$ Eigenvalue:

$$
\begin{aligned}
|A - \lambda I| &= \begin{bmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{bmatrix} \\
&= (1-\lambda)(-7-\lambda) + 16 = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2 = 0 \\
&\Rightarrow\ \lambda_1 = \lambda_2 = -3
\end{aligned}
$$

Eigenvectors: Find x such that $Ax = \lambda x$

$$(A - \lambda)x = (A+3)x = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$4x_1 - 4x_2 = 0, \quad 4x_1 - 4_x 2 = 0\ \Rightarrow x_1 = x_2$$
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2$$

So the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Example 3 $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ Eigenvalue:

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - 4 = \lambda^2 - 4\lambda - 1$$
$$\Rightarrow\ \lambda_1 = 2 + \sqrt{5}, \quad \lambda_2 = 2 - \sqrt{5}$$

Eigenvectors: Find x such that $Ax = \lambda x$

$$(A - \lambda_1)x = (A - (2+\sqrt{5})x = \begin{bmatrix} -(1+\sqrt{5}) & 2 \\ 2 & 1-\sqrt{5} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-(1+\sqrt{5})x_1 + 2x_2 = 0 \;\Rightarrow x_2 = \left(\frac{1+\sqrt{5}}{2}\right) x_1$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \left(\frac{1+\sqrt{5}}{2}\right) x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} x_1$$

So the eigenvector corresponding to $\lambda_1 = 2 + \sqrt{5}$ is $\begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}$

$$(A - \lambda_2)x = (A - (2-\sqrt{5})x = \begin{bmatrix} -1+\sqrt{5} & 2 \\ 2 & 1+\sqrt{5} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow\; (\sqrt{5}-1)x_1 + 2x_2 = 0 \;\Rightarrow\; x_2 = \frac{1-\sqrt{5}}{2}$$

So the eigenvector corresponding to $\lambda_2 = 2 - \sqrt{5}$ is $\begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}$

5.3 Basic Theory of Systems of first order linear equations

- **General form**:

$$x' = p(t)x + g(t)$$

$$x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad p(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \vdots & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix}, \quad g(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

  Homogeneous System: ($g(t) = 0$)

$$x' = p(t)x$$

- **Superposition Principle**: If the vector functions $x^{(1)} = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}$ and $x^{(2)} = \begin{bmatrix} x_{12}(t) \\ \vdots \\ x_{n2}(t) \end{bmatrix}$ are two solutions of $x' = p(t)x$, then the linear combinations $C_1x^{(1)} + C_2x^{(2)}$ is also a solution for any constants C_1 and C_2.

Example 1: $x' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x$

$x^{(1)} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}, \quad x^{(2)} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$

$x = C_1 x^{(1)} + C_2 x^{(2)} = \begin{bmatrix} C_1 e^{3t} + C_2 e^{-t} \\ 2C_1 e^{3t} - 2C_2 e^{-t} \end{bmatrix}$

- **The Wronskian**

Consider n solutions of $x' = p(t)x$:

$$x^{(1)} = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \cdots, x^{(n)} = \begin{bmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}.$$

Define $\mathrm{X}(t) = \begin{bmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{bmatrix} = [x^{(1)}, x^{(2)}, \cdots, x^{(n)}]$

The Wronskian of these n solutions is defined by $\boxed{W[x^{(1)}, \cdots, x^{(n)}] = \det(\mathrm{X}(t))}$

Example 1(continued)

$\mathrm{X}(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$

$W[x^{(1)}, x^{(2)}] = -4e^{2t}$

Relation between the Wronskian of linear system and that of second-order linear equation:

Linear equation: $y'' + p(t)y' + q(t)y = 0$

Solutions: $y_1(t)$, $y_2(t)$

Wronskian: $W[y_1, y_2] = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y'_(t) & y'_2(t) \end{bmatrix}$

Transform to linear system: $y'' + p(t)y' + q(t)y \xLeftrightarrow{x_1=y, x_2=y'} \begin{cases} x'_1 = x_2 \\ x'_2 = -q(t)x_1 - p(t)x_2 \end{cases}$

$y_1(t)$ solution $\Leftrightarrow x^{(1)} = \begin{bmatrix} y_1(t) \\ y'_1(t) \end{bmatrix}$ solution

$y_2(t)$ solution $\Leftrightarrow x^{(2)} = \begin{bmatrix} y_2(t) \\ y'_2(t) \end{bmatrix}$ solution

$W[y_1, y_2] = W[x^{(1)}, x^{(2)}] = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{bmatrix}$ Same!

General Solution: Suppose that vector functions $x^{(1)}, \cdots, x^{(n)}$ are n solutions of $x' = p(t)x$, $\alpha \le t \le \beta$. If there exists a point $t_0 \in [\alpha, \beta]$ such that

$$W[x^{(1)}, \cdots, x^{(n)}](t_0) = \det(\mathrm{X}(t_0)) \neq 0,$$

then the general solution of $x' = p(t)x$ is

$$x = \phi(t) = C_1 x^{(1)}(t) + C_2 x^{(2)}(t) + \cdots + C_n x^{(n)}(t) = \mathrm{X}(t)C$$

where $C = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$.

Example 2 $x' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x$

$x^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t, \quad x^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}, \quad x^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t}$

$x = \begin{bmatrix} C_1 e^t \\ C_2 e^{2t} \\ C_3 e^{3t} \end{bmatrix}$ general solution.

5.4 Homogeneous linear systems with Constant Coefficients

Consider $x' = Ax$, A: constant matrix of $n \times n$.

Seek the solution of the form

$$x = \xi e^{rt}$$

where r and the constant vector ξ are to be determined.

$x' = Ax \;\Rightarrow\; r\xi e^{rt} = A\xi e^{rt}$

$A\xi = r\xi$ or $(A - rI)\xi = 0$

This implies that if r and ξ are an eigenpair of A, then $x(t) = \xi e^{rt}$ is a solution of $x' = Ax$. If A has n eigenvectors:

$$A\xi^{(1)} = \lambda_1 \xi^{(1)}, \cdots, A\xi^{(n)} = \lambda_n \xi^{(n)}$$

and $W[\xi^{(1)}, \cdots, \xi^{(n)}] \neq 0$, then the general solution is

$$x = \phi(t) = C_1 \xi^{(1)} e^{\lambda_1 t} + \cdots + C_n \xi^{(n)} e^{\lambda_n t}$$

Examples

- $A = \text{diag}(2, -3, 4)$

  Eigenvalues are 2, -3, 4, corresponding eigenvectors of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$

  $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 - 9 = \lambda^2 - 3\lambda - 7$

  $\lambda_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{3 \pm \sqrt{37}}{2} \Rightarrow \lambda_1 = -1.5414$, $\lambda_2 = 4.5414$.
  Since $A\xi = \lambda\xi_1 \Rightarrow (A - \lambda I)\xi = 0$

  $\lambda_1 = -1.5414$: $\begin{bmatrix} 1-(-1.5414) & 3 \\ 3 & 2-(-1.5414) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  $2.5414x_1 + 3x_2 = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -\dfrac{2.5414}{3}x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.8471 \end{bmatrix} x_1$

  So $\xi_1 = \begin{bmatrix} 1 \\ -0.8471 \end{bmatrix}$.

  $\lambda_2 = 4.5414$: $\begin{bmatrix} 1-4.5414 & 3 \\ 3 & 2-4.5414 \end{bmatrix} = \begin{bmatrix} -3.5414 & 3 \\ 3 & -2.5414 \end{bmatrix}$

  $\xi_2 = \begin{bmatrix} 3 \\ 3.5414 \end{bmatrix}$

  $x = C_1 \begin{bmatrix} 1 \\ -0.8471 \end{bmatrix} e^{-1.5414t} + C_2 \begin{bmatrix} 3 \\ 3.5414 \end{bmatrix} e^{4.5414t}$.

- $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

  $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$

  $\lambda_1 = 3$, $\lambda = -1$.

  $A - \lambda_1 I = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \Rightarrow \xi_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  $A - \lambda_2 I = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \Rightarrow \xi_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

  $x = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t}$.

Summary: $x' = Ax$

- case 1: A Hermitian: $a_{ij} = \overline{a_{ji}}$. Every eigenvalues or a Hermitian matrix are real, and all the eigenvectors are linear independent.

$$A\xi^{(i)} = r_i \xi^{(i)}, \quad i = 1, \cdots, n, \quad W[\xi^{(i)}, \cdots, \xi^{(n)}] \neq 0$$

$$x^{(1)} = \xi^{(1)} e^{r_1 t}, \cdots, x^{(n)} = \xi^{(n)} e^{r_n t}$$

$$x = C_1 \xi^{(1)} e^{r_1 t} + \cdots + C_n \xi^{(n)} e^{r_n t}$$

- case 2: A non-Hermitian, that is $A* \neq A$, A real.
  - (i) All eigenvalues are real and distinct.

$$A\xi^{(i)} = r_i \xi^{(i)}, \quad i = 1, \cdots, n$$

$$x = C_1 \xi^{(1)} e^{r_1 t} + \cdots + C_n \xi^{(n)} e^{r_n t}$$

  - (ii) Some eigenvalues occur in complex conjugate pairs, others are real and distinct.
  - (iii) some eigenvalues are repeated; others are real and distinct.

Example 1 $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2)$$

$\lambda_1 = 8 \quad \lambda_2 = 2$

$$A - \lambda_1 I = \begin{bmatrix} 5-8 & 3 \\ 3 & 5-8 \end{bmatrix} \Rightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} 5-2 & 3 \\ 3 & 5-2 \end{bmatrix} \Rightarrow \xi^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Example 2 $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 1+i \\ 1-i & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - (1+i)(1-i) = \lambda^2 - 3\lambda + 2 - (1-i^2) = \lambda^2 - 3\lambda + 2 - 2 = \lambda(\lambda - 3)$$

$\lambda_1 = 0$ and $\lambda_2 = 3$.

$$A - \lambda_1 I = A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \Rightarrow \xi^{(1)} = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} -2 & 1+i \\ 1-i & -1 \end{bmatrix} \Rightarrow \xi^{(2)} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

$$x = C_1 \begin{bmatrix} 1+i \\ -1 \end{bmatrix} e^{0t} + C_2 \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{3t} = C_1 \begin{bmatrix} 1+i \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{3t}$$

5.5 Complex Eigenvalues

To solve $x' = Ax$ (*)

A is a real matrix and has a complex eigenvalue. $r_1 = \lambda + i\mu$:

$A\xi^{(1)} = r_1\xi^{(1)} \Rightarrow \overline{A\xi^{(1)}} = \overline{r_1\xi^{(1)}} \Rightarrow A\bar{\xi}^{(1)} = \bar{r}_1 \cdot \bar{\xi}^{(1)}$

$\Rightarrow \bar{r}_1$ is also an eigenvalue and the corresponding eigenvector is $\bar{\xi}^{(1)}$.

$x^{(1)} = \xi^{(1)}e^{r_1 t}, \quad x^{(2)} = \bar{\xi}^{(1)}e^{\bar{r}_1 t}$

$C_1x^{(1)} + C_2x^{(2)} = C_1\xi^{(1)}e^{r_1 t} + C_2\bar{\xi}^{(1)}e^{\bar{r}_1 t} = C_1\xi^{(1)}e^{r_1 t} + C_2\overline{\xi^{(1)}e^{r_1}}$

Let $\xi^{(1)} = a + ib, \quad r_1 = \lambda + i\mu, \quad e^{r_1 t} = e^{\lambda t}[\cos\mu t + i\sin\mu t]$, where a and b are real vectors.

$\xi^{(1)}e^{r_1 t} = (a+ib)e^{\lambda t}(\cos\mu t + i\sin\mu t) = e^{\lambda t}[a\cos\mu t + i(b\cos\mu t + a\sin\mu t) - b\sin\mu t]$

$\bar{\xi}^{(1)} = a - ib, \quad \bar{r}_1 = \lambda - i\mu, \quad e^{\bar{r}_1 t} = e^{\lambda t}[\cos\mu t - i\sin\mu t]$

$\bar{\xi}^{(1)}e^{\bar{r}_1 t} = (a-ib)e^{\lambda t}(\cos\mu t - i\sin\mu t) = e^{\lambda t}[a\cos\mu t - i(b\cos\mu t + a\sin\mu t) - b\sin\mu t]$

if $C_1 = C_2 = \dfrac{1}{2}$: $C_1x^{(1)} + C_2^{(2)} = \text{Re}[\xi^{(1)}e^{r_1 t}] = \boxed{e^{r_1 t}[a\cos\mu t - b\sin\mu t]} \leftarrow$ a real solution of $(*)$.

if $C_1 = \dfrac{1}{2i}, C_2 = -\dfrac{1}{2i}$: $C_1x^{(1)} + C_2x^{(2)} = \text{Im}[\xi^{(1)}e^{r_1 t}] = \boxed{e^{\lambda t}[a\sin\mu t + b\cos\mu t]} \leftarrow$ another real solution of $(*)$.

Summary

$$r_1 = \lambda + i\mu, \quad \xi^{(1)} = a + bi$$
$$\bar{r}_1 = \lambda - i\mu, \quad \bar{\xi}^{(1)} = a - bi$$

Two real solutions are

$$e^{\lambda t}[a\cos\mu t - b\sin\mu t], \quad e^{rt}[a\sin\mu t + b\cos\mu t]$$

If r_3,...,r_n real and distinct, the general solution

$$x = C_1e^{\lambda t}[a\cos\mu t - b\sin\mu t] + C_2e^{rt}[a\sin\mu t + b\cos\mu t] + C_3\xi^{(3)}e^{r_3 t} + \cdots + C_n\xi^{(n)}e^{r_n t}$$

Example 1: $x' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} x$

$|A - rI| = (-\frac{1}{2} - r)^2 + 1 = 0 \;\Rightarrow\; -\frac{1}{2} - r = \pm i$

$r_1 = -\frac{1}{2} + i, \quad r_2 = -\frac{1}{2} - i = \bar{r}_1$

$A - r_1 I = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$

$\xi^{(1)} = \begin{bmatrix} -1 \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{a} + i \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b}$

$x = C_1 e^{-\frac{1}{2}t}(\begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t) + C_2 e^{-\frac{1}{2}t}(\begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t)$

Example 2: $x' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$

$|A - rI| = (-1 - r)^2 + 4 = 0 \;\Rightarrow\; (-1 - r)^2 = -4 \;\Rightarrow\; -1 - r = \pm 2i$

$r_1 = -1 + 2i, \quad r_2 = -1 - 2i \;\Rightarrow \lambda = -1 \quad \mu = 2$

$A - r_1 I = \begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \;\Rightarrow\; \xi^{(1)} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$

$\xi^{(1)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$x = C_1 e^{-t}(\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin 2t) + C_2 e^{-t}(\begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin 2t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos 2t)$

5.6 Repeated Eigenvalues

$$x' = Ax$$

If $r = r_1$ is a k-fold root ($r_1 = r_2 = \cdots = r_k$) of the equation

$$\det(A - rI) = 0$$

then r_1 is said to be an eigenvalue of multiplicity k of A.

Now we assume that $r = r_1$ is an eigenvalue of multiplicity 2. Suppose

$$A\xi^{(1)} = r_1 \xi^{(1)} \quad (\xi^{(1)} \neq 0)$$

$x^{(1)} = \xi^{(1)} e^{r_1 t}$ is a solution of $x' = Ax$. Seek another solution in the form:

$$x^{(2)} = \xi^{(1)} t e^{r_1 t}$$

However, by $\frac{dx^{(2)}}{dt} = Ax^{(2)}$

$$\xi^{(1)} e^{r_1 t} + \xi^{(1)} t e^{r_1 t} = A\xi^{(1)} t e^{r_1 t}$$
$$\text{or } \xi^{(1)} e^{r_1 t} + \xi^{(1)} t e^{r_1 t} = r_1 \xi^{(1)} t e^{r_1 t}$$

$$\Rightarrow \ \xi^{(1)} e^{r_1 t} = 0 \ \Rightarrow \ \xi^{(1)} = 0 \text{ (conflict)}$$

Seek another solution in the form:

$$x^{(2)} = \xi^{(1)} t e^{r_1 t} + \eta e^{r_1 t}$$

By $\frac{dx^{(2)}}{dt} = Ax^{(2)}$,

$$\xi^{(1)} e^{r_1 t} + \xi^{(1)} t r_1 e^{r_1 t} + \eta r_1 e^{r_1 t} = r_1 \xi^{(1)} t e^{r_1 t} + A\eta e^{r_1 t}$$
$$\Rightarrow \ (A - r_1 I)\eta = \xi^{(1)} \ (*)$$

since $\det(A - r_1 I) = 0$. (*) may not have solution

<u>**General solution**</u>: $\boxed{x = C_1 \xi^{(1)} e^{r_1 t} + C_2 [t\xi^{(1)} e^{r_1 t} + \eta e^{r_1 t}]}$

Example 1 $x' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$|A - rI| = (4 - r)(-4 - r) + 16 = r^2 - 16 + 16 = r^2 = 0 \ \Rightarrow \ r_1 = r_2 = 0$

$A - r_1 I = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \Rightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$(A - 0I)\eta = \xi^{(1)}$

$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{cases} 4\eta_1 - 2\eta_2 = 1 \\ 8\eta_1 - 4\eta_2 = 2 \end{cases}$

$4\eta_1 - 2\eta_2 = 1 \ \Rightarrow$ one such solution is $\eta = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$

$x^{(1)} = \xi^{(1)} e^{0t} = \xi^{(1)}$

$x^{(2)} = \xi^{(1)} t e^{0t} + \eta e^{0t} = \xi^{(1)} t + \eta = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$

$x = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \left(t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} \right)$

By $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{cases} C_1 + C_2 = 1 \\ 2C_1 + \frac{3}{2}C_2 = 0 \end{cases} \Rightarrow \; C_1 = -3, \quad C_2 = 4$$

$$x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Example 2: $x' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} x$

$$|A - rI| = (1-r)(3-r) + 1 = r^2 - 4r + 4 = (r-2)^2 = 0 \;\Rightarrow\; r_1 = r_2 = 2$$

$$A - 2I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \;\Rightarrow\; \xi^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A - 2I)\eta = \xi^{(1)} \;\Rightarrow\; \begin{cases} -\eta_1 - \eta_2 = 1 \\ \eta_1 + \eta_2 = -1 \end{cases}$$

$\eta_1 + \eta_2 = -1 \;\Rightarrow\;$ one possible solution is $\eta = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$x = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + C_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2t} \right)$$

Review of Chapter 7 System of First-Order Linear Equation

$$x' = P(t)x + g(t)$$
$$x' = Ax, \quad A = (a_{ij})_{n\times n}, \quad a_{ij} \text{ constant.}$$

Section 7.4 Homogeneous system $x' = p(t)x$:

- Superposition Theory.
- The Wronskian.
- General solutions

Section 7.5 Homogeneous system with constant coefficients $x' = Ax$:

- case 1: A Hermitian: $a_{ij} = \overline{a_{ji}}$. Every eigenvalues or a Hermitian matrix are real, and all the eigenvectors are linear independent.

$$A\xi^{(i)} = r_i\xi^{(i)}, \quad i = 1, \cdots, n, \quad W[\xi^{(i)}, \cdots, \xi^{(n)}] \neq 0$$

$$x^{(1)} = \xi^{(1)}e^{r_1 t}, \cdots, x^{(n)} = \xi^{(n)}e^{r_n t}$$

$$x = C_1\xi^{(1)}e^{r_1 t} + \cdots + C_n\xi^{(n)}e^{r_n t}$$

- case 2: A non-Hermitian, that is $A* \neq A$, A real.

  (i) All eigenvalues are real and distinct.

$$A\xi^{(i)} = r_i\xi^{(i)}, \quad i = 1, \cdots, n, \quad W[\xi^{(1)}, \cdots, \xi^{(n)}] \neq 0$$

$$x = C_1\xi^{(1)}e^{r_1 t} + \cdots + C_n\xi^{(n)}e^{r_n t}$$

  (ii) Some eigenvalues occur in complex conjugate pairs, others are real and distinct.
  The simplest case is that:
  $r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, r_3,...,r_n real and distinct. Then we can find n eigenvectors $\xi^{(1)}, \cdots, \xi^{(n)}$ such that

$$A\xi^{(1)} = r_1\xi^{(1)}, \quad A\xi^{(2)} = r_2\xi^{(2)}$$
$$A\xi^{(3)} = r_3\xi^{(3)}, \cdots, A\xi^{(n)} = r_n\xi^{(n)}$$

  where $\xi^{(1)}$ and $\xi^{(2)}$ are complex and $\xi^{(3)}$,...,$\xi^{(n)}$ are real.
  Let $\xi^{(1)} = a + ib$, then $\xi^{(2)} = a - ib$.

$$x(t) = C_1e^{\lambda t}(a\cos\mu t - b\sin\mu t) + C_2e^{\lambda t}(a\sin\mu t + b\cos\mu t)$$
$$+ C_3\xi^{(3)}e^{r_3 t} + \cdots + C_n\xi^{(n)}e^{r_n t}$$

  (iii) some eigenvalues are repeated; others are real and distinct.
  The simplest case is $r_1 = r_2$ real, and $r_3, \cdots, r_n$ real and distinct. Then the general solution

$$x = C_1\xi^{(1)}e^{r_1 t} + C_2[t\xi^{(1)}e^{r_1 t} + \eta e^{r_1 t}] + C_3\xi^{(3)}e^{r_3 t} + \cdots + C_n\xi^{(n)}e^{r_n t}$$

  where η is given by $(A - r_1 I)\eta = \xi^{(1)}$. Since det$(A - r_1 I) = 0$, η may not exist.

Chapter 4

Higher Order Linear Equations

The thoeretical structure and solution methods developed for 2nd order linear DEs can be extended to 3rd order and higher order linear DEs.

4.1 General theory of nth order linear equations

General forms:

$$L[y] = \frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}\frac{dy}{dt} + P_n y = g(t)$$

or

$$L[y] = y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_{n-1}y' + P_n y = g(t)$$

n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \cdots, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

<u>Existence Theorem</u>:

If the functions P_1,P_2,...,P_n, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the IVP. The solution is valid throughout the interval I.

Assume n solutions y_1, y_2,...,y_n are known for the homogeneous equation

$$y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_{n-1}y' + P_n y = 0$$

General soln:

$$y(t) = C_1 y_1(t) + \cdots + C_n y_n(t) \quad (*)$$

Solutions for IVP satisfies the ICs, i.e.

$$\begin{cases} C_1 y_1(t_0) + \cdots + C_n y_n(t_0) = y_0 \\ C_1 y_1'(t_0) + \cdots + C_n y_n'(t_0) = y_0' \\ \cdots C_1 y_1^{(n-1)}(t_0) + \cdots + C_n y_n^{(n-1)}(t_0) = y_0^{(n-1)} \end{cases}$$

there is a solution C_1,...,C_n of this linear system if and only if

$$W[y_1, y_2, \cdots, y_n](t) = \begin{vmatrix} y_1(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{vmatrix} \neq 0$$

Thus, (*) is the general solution $\Longleftrightarrow$ there exists a point to t_0 such that $W[y_1, ..., y_n](t_0) \neq 0$.

<u>The general solution of the inhomogeneous equation:</u>

$$y = C_1 y_1(t) + \cdots + C_n y_n(t) + Y(t)$$

where $y_1(t), \cdots, y_n(t)$ are linearly independent solutions of the homogeneous equation and $Y(t)$ is a particular solution of the inhomogeneous equations.

<u>Linear dependence</u> of n functions:

There exist n not all zero constants $k_1, \cdots, k_n$ such that

$$k_1 y_1(t) + \cdots + k_n y_n(t) = 0, \quad \text{for all } t \in I$$

Example 1 existence interval:

$$ty''' + \sin(t)y'' + 3y = \cos t$$

Soln: divide t from the equation, we have $y''' + \frac{\sin t}{t}y'' + \frac{3}{t}y = \frac{\cos t}{t}$. $\frac{\sin t}{t}$, $\frac{3}{t}$ and $\frac{\cos t}{t}$ are continuous on $(-\infty, 0) \cup (0, \infty)$, which is the existence interval.

Example 2 linear dependence:

$$f_1(t) = 2t - 3, \quad , f_2(t) = t^2 + 1, \quad f_3 = 2t^2 - 1$$

	f_1	f_2	f_3
t^2	0	1	2
t	2	0	0
1	-3	1	-1

$$\begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ -3 & 1 & -1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = 6$$

So they are linear independent.

<u>Abel's theorem</u>:

$$y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = 0$$

then

$$W[y_1, ..., y_n](t) = Ce^{-\int P_1(t)dt}$$

Example 3 Find the Wronskian of 1, $\cos t$, $\sin t$ and from DE and Abel's theorem

$$y''' + y' = 0$$

$$\begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = \sin^2 t + \cos^2 t = 1 \neq 0$$

Easy to check that 1, $\cos t$, $\sin t$ are solutions of $y''' + y' = 0$. According to Abel's theorem,

$$W[1, \cos t, \sin t](t) = Ce^{-\int 0dt} = C \cdot e^{0+C_1} \neq 0.$$

4.2 Homogeneous Equations with Constant Coefficients

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

Look for solution in form of $y = e^{rt}$.

Characteristic equation:

$$a_0 r^n + a_1 r^{n-1} + \cdots a_{n-1} r + a_n = 0$$

- case 1: Real and unequal roots: r_1, r_2,...,r_n. General solutions:

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \cdots + C_n e^{r_n t}$$

- case 2: Complex roots:

$$r_1 = \lambda + i\mu \;\Rightarrow\; \bar{r}_1 = \lambda - i\mu \text{ is also a rot}$$
$$y_1(t) = e^{\lambda t} \cos \mu t \quad y_2(t) = e^{\lambda} \sin \mu t$$

- case 3: repeated roots. r_1 is real root with multiplicity $s(\leq n)$.

$$\Rightarrow\; e^{r_1 t},\; te^{r_1 t},\; t^2 e^{r_1 t}, \cdots, t^{s-1} e^{r_1 t} \text{ roots}$$

  $r_1 = \lambda + i\mu$ complex root with multiplicity s:

$$e^{\lambda t} \cos \mu t,\; te^{\lambda t} \cos \mu t, \cdots, t^{s-1} e^{\lambda t} \cos \mu t$$
$$e^{\lambda t} \sin \mu t,\; te^{\lambda t} \sin \mu t, \cdots, t^{s-1} e^{\lambda t} \sin \mu t$$

Example 1: $y^{(4)} + y = 0$.

$r^4 + 1 = 0, \;\Rightarrow\; r^4 = (-1)Re^{i\theta}, \quad R = 1,\; \theta = \pi + 2m\pi$

$r^4 = e^{i(\pi + 2m\pi)} \;\Rightarrow\; r = e^{i(\pi+2m\pi)/4}, \quad m = 0, 1, 2, 3, \cdots$

$r = e^{i\frac{\pi}{4}},\; e^{i\frac{3\pi}{4}},\; e^{i\frac{5\pi}{4}},\; e^{i\frac{7\pi}{4}}$

$r = \dfrac{1+i}{\sqrt{2}}, \dfrac{-1+i}{\sqrt{2}}, \dfrac{-1-i}{\sqrt{2}}, \dfrac{1-i}{\sqrt{2}}$

$y(t) = e^{\frac{t}{\sqrt{2}}} \left[C_1 \cos \dfrac{t}{\sqrt{2}} + C_2 \sin \dfrac{t}{\sqrt{2}} \right] + e^{-\frac{t}{\sqrt{2}}} \left[C_3 \cos \dfrac{t}{\sqrt{2}} + C_4 \sin \dfrac{t}{\sqrt{2}} \right]$

Example 2: $y^{(4)} - y'' = 0$

$r^4 - r^2 = 0 \quad r^2(r^2-1) = 0 \quad r^2(r-1)(r+1) = 0$
$r_1 = r_2 = 0, \quad r_3 = 1, \quad r_4 = -1$
$y = C_1 e^{0t} + C_2 t e^{0t} + C_3 e^t + C_4 e^{-t}$
$= C_1 + C_2 t + C_3 e^t + C_4 e^{-t}$

Example 3: $y''' - 3y'' + 3y' - y = 0$

$r^3 - 3r^2 + 3r - 1 = 0 \quad r^3 - 1 - 3r^2 + 3r = 0$
$(r-1)(r^2+r+1) - 3r(r-1) = 0 \quad (r-1)(r^2-2r+1) = 0$
$(r-1)(r-1)^2 = 0 \quad (r-1)^3 = 0$
$r_1 = r_2 = r_3 = 1$
$y = C_1 e^t + C_2 t e^t + C_3 t^2 e^t$

4.3 The method of undetermined coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t)$$

Example 1: $y''' - 3y'' + 3y' - y = 6e^t$

$y''' - 3y'' + 3y' - y = 0 \quad r^3 - 3r^2 + 3r - 1 = 0 \quad (r-1)^3 = 0$
$y(t) = C_1 e^t + C_2 t e^t + C_3 t^2 e^t + Y(t)$
$\boxed{Y(t) = At^3 e^t}$ A to be determined
$Y' = 3At^2 e^t + At^3 e^t$
$Y'' = 6Ate^t + 3At^2 e^t + 3At^2 e^t + At^3 e^t = 6Ate^t + 6At^2 e^t + At^3 e^t$
$Y''' = 6Ae^t + 18Ate^t + 9At^2 e^t + At^3 e^t$

Since for $y = e^t,\ te^t,\ t^2 e^t$, $y''' - 3y'' + 3y' - 4 = 0$, it must be $6Ae^t = 6e^t \Rightarrow A = 1$.

$y(t) = C_1 e^t + C_2 t e^t + C_3 t^2 e^t + t^3 e^t$.

Example 2: $y''' - 4y' = t + 3\cos t + e^{-2t}$

$y''' - 4y' = 0, \quad r^3 - 4r = 0, \quad r_1 = 0,\ r_2 = 2,\ r_3 = -2$
$y(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t} + Y(t)$
$y''' - 4y' = t:\ Y_1(t) = t(A_0 t + A_1)$
$y''' - 4y' = 3\cos t:\ Y_2(t) = B\cos t + C\sin t$
$Y''' - 4y' = e^{-2t}:\ Y_3(t) = Ete^{-2t}$
$Y_1''' - 4Y_1' = -(2A_0 t + 4A_1) \ \Rightarrow\ A_0 = -\dfrac{1}{2},\ A_1 = 0$
$Y_2''' - 4Y_2' = B\sin t - C\cos t + 4B\sin t - 4C\cos t = 5B\sin t - 5C\cos t = 3\cos t \ \Rightarrow\ B = 0,\ C = -\dfrac{3}{5}$
$Y_3''' - 4Y_3' = 8Ee^{-2t} = e^{-2t} \ \Rightarrow\ E = \dfrac{1}{8}$
$Y = -\dfrac{1}{2}t^2 - \dfrac{3}{5}\sin t + \dfrac{1}{8}te^{-2t}$

Example 3: $y''' - y' = te^{-t} + 2\cos t$

$r^3 - r = 0, \quad (r-1)(r^2 + r + 1) = 0, \quad r_1 = 1,\ r_2\ r_3$ complex
$r_2,\ r_3:\ \lambda = -\dfrac{1}{2} \quad \mu = \dfrac{\sqrt{3}}{2}$
$y(t) = C_1 e^t + e^{-1/2t}\left(C_2 \cos\dfrac{\sqrt{3}}{2}t + C_3 \sin\dfrac{\sqrt{3}}{2}t\right) + Y(t)$
$y''' - y' = te^{-t}:\ Y_1(t) = (A_0 + A_1 t)te^{-t}$
$y''' - y' = 2\cos t:\ Y_2(t) = B_0 \cos t + B_1 \sin t$
$Y_1''' - Y_1' = 4A_1 te^{-t} + (2A_0 - 6A_1)e^{-t}, \quad A_1 = \dfrac{1}{4},\ A_0 = \dfrac{3}{4}$
$Y_2''' - Y_2' = 2B_0 \sin t - 2B_1 \cos t, \quad B_0 = 0,\ B_1 = -1$
$Y = \left(\dfrac{3}{4} + \dfrac{1}{4}t\right)te^{-t} - \sin t$

4.4 The method of variation of parameters

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

solution to homogeneous equation

$$y_c(t) = C_1 y_1(t) + \cdots + C_n y_n(t)$$

Look for

$$Y(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t)$$

Wronskian

$$W[y_1, \cdots, y_n](t) = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Replace the <u>mth</u> column of $W[y_1, \cdots, y_n](t)$ with $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

$$\downarrow$$

$$W_m(t) = \begin{vmatrix} y_1 & \cdots & y_{m-1} & 0 & y_{m+1} & \cdots & y_n \\ y_1' & \cdots & y_{m-1}' & 0 & y_{m+1}' & \cdots & y_n' \\ & & & \cdots & & & \\ y_1^{(n-1)} & \cdots & y_{m-1}^{(n-1)} & 1 & y_{m+1}^{n-1} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$Y(t) = \sum_{m=1}^{N} y_m(t) \int_t^{t_0} \frac{g(s)W_m(s)}{W(s)} ds$$

Example 1: $y''' - y'' - y' + y = t$, $y_1 = e^t$, $y_2 = te^t$, $y_3 = e^{-t}$. Find $Y(t)$.

$$W(t) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & e^t + te^t & -e^{-t} \\ e^t & 2e^t + te^t & e^{-t} \end{vmatrix} = 4e^t.$$

$$W_1(t) = \begin{vmatrix} 0 & te^t & e^{-t} \\ 0 & e^t + te^t & -e^{-t} \\ 1 & 2e^t + te^t & e^{-t} \end{vmatrix} = -2t - 1.$$

$$W_2(t) = \begin{vmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & e^{-t} \end{vmatrix} = 2.$$

$$W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & e^t + te^t & 0 \\ e^t & 2e^t + te^t & 1 \end{vmatrix} = e^{2t}.$$

$$\int \frac{t(-2t-1)}{4e^t} dt = \frac{1}{4}(t^2 + 5t + 5)e^{-t}, \quad y_1(t) \cdot \frac{1}{4}(t^2 + 5t + 5)e^{-t} = \frac{1}{4}(t^2 + 5t + 5)$$

$$\int \frac{2t}{4e^t} dt = -\frac{te^{-t} + e^{-t}}{2}, \quad y_2(t) \cdot -\frac{te^{-t} + e^{-t}}{2} = -\frac{t^2 + t}{2}$$

$$\int \frac{te^{2t}}{4e^t} dt = \frac{te^t - e^t}{2}, \quad y_3(t) \cdot \frac{te^t - e^t}{2} = \frac{t-1}{2}$$

$Y(t) = \dfrac{-t^2 + 5t - 3}{4}$

Example 2 $y''' + y' = \tan t$

$r^3 + r = 0 \quad r(r^2+1) = 0 \quad r_1 = 0, \quad \mu = 1$

$y_1 = 1, \quad y_2 = \cos t, \quad y_3 = \sin t$

$$W(t) = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & \sin t \end{vmatrix} = \cos^t - \sin^2 t$$

$$W_1(t) = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & \sin t \end{vmatrix} = 1$$

$$W_2(t) = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & \sin t \end{vmatrix} = -\cos t$$

$$W_3(t) = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -\sin t$$

$$\int \frac{\tan t * 1}{\cos^2 t - \sin^2 t} dt = \int \frac{\sin t}{\cos t(2\cos^2 - 1)} dt = -\int \frac{1}{\cos t(2\cos^2 - 1)} d\cos t \overset{u=\cos t}{=} -\int \frac{1}{u(2u^2-1)} du$$

$$-\int \frac{1}{u(2u^2-1)} du = \ln u - \frac{1}{2}\ln(2u^2-1) \overset{\cos t = u}{=} \ln(\cos t) - \frac{1}{2}\ln(\cos 2t)$$

$$\int \frac{\tan t(-\cos t)}{2\cos^2 t - 1} dt = \int \frac{d\cos t}{2\cos^2 t - 1} = \int \frac{1}{2\cos^2 u - 1} du = -\frac{\sqrt{2}}{2}\text{arctanh}(\sqrt{2}t)$$

Chapter 4

The Laplace Transform

Integral transforms are a class of most useful tools for solving linear differential equations. We are going to study one of these transforms: the Laplace transform.

4.1 Definition of the Laplace transform

Consider a function $f(t)$ defined for $t \geq 0$. Define a new function

$$L\{f(t)\} \equiv F(s) = \int_0^\infty e^{st} f(t)dt$$

Improper integrals:

$$\int_a^\infty f(t)dt = \lim_{A\to\infty} \int_a^A f(t)dt$$

divergence, convergence, existence

Examples:

- $f(t) = e^{ct}$
  $\displaystyle\int_0^\infty e^{ct}dt = \frac{1}{c}e^{ct}\Big|_0^\infty$.
  $c > 0$: $e^{c\cdot\infty} = \infty$, divergent, so the improper integral does not exist.
  $c < 0$: $e^{c\cdot\infty} = 0$, convergence, so $\displaystyle\int_0^\infty e^{ct}dt = -\frac{1}{c}$.
- $f(t) = \dfrac{1}{t}$
  $\displaystyle\int_0^\infty \frac{1}{t}dt = \ln t|_0^\infty$
  $\ln 0 = -\infty$, $\ln\infty = \infty$. divergent.

- $f(t) = \dfrac{1}{t^2}$.

  $\displaystyle\int_0^\infty \frac{1}{t^2} dt = -t^{-1}|_0^\infty$

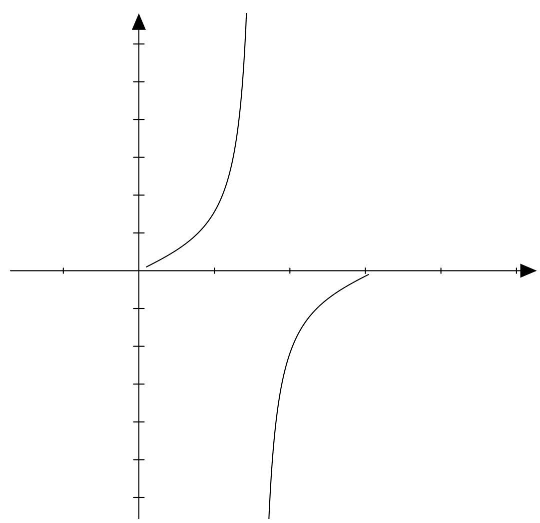
  $\dfrac{1}{0} = \infty$, $\dfrac{1}{\infty} = 0$. divergent.

<u>Piecewise Continuous</u>: A function is said to be piecewise continuous on an interval $\alpha \leq t \leq \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n \leq \beta$ such that

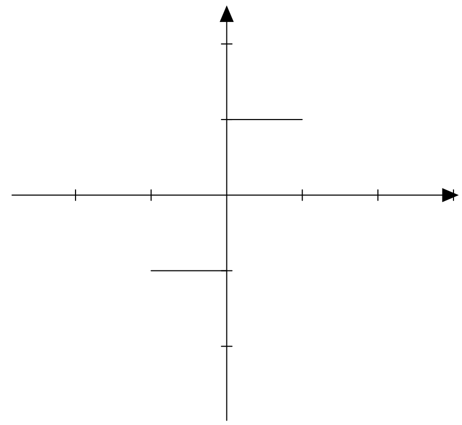
1. f is continuous on each open subinterval $t_{i-1} < t < t_i$ $(i = 1, \cdots, n)$.
2. f approaches a finite limit as the end points of each subinterval are approached from within the subinterval.

Examples:

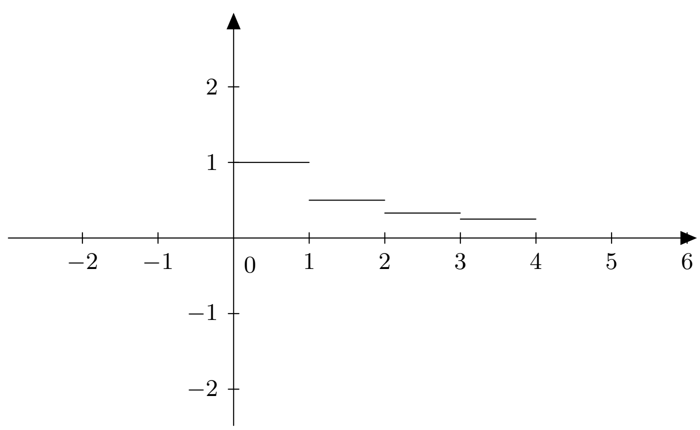
- $f(t) = \tan t, \quad 0 \leq t \leq \pi$



- $f(t) = \begin{cases} -1, & \text{for } t \in [-1, 0) \\ 1, & \text{for } t \in [0, 1] \end{cases}$



- $f(t) = \begin{cases} 1, & \text{for } t \in [0,1) \\ \frac{1}{2}, & \text{for } t \in [1,2) \\ \cdots \\ \frac{1}{n}, & \text{for } t \in [n-1,n) \\ \cdots \end{cases}$



Existence of integrals

Property 1 If f is piecewise continuous on an interval $a \le t \le A$ then $\int_a^A f(t)dt$ exists.

Property 2 If f is piecewise continuous for $t \ge a$ (or $t \le [a, \infty)$), then $\int_a^A f(t)dt$ exists for each $A > a$.

Example $\int_a^\infty f(t)dt$ exists if $\int_a^A f(t)$ exists?

$$f(t) = \frac{1}{t}, \quad t \ge 1, \quad \int_1^\infty \frac{1}{t}dt = \ln(\infty) - \ln 1 = \infty.$$

Property 3 (Comparison Theorem, 6.1.1) Consider a function f defined $t \geq a$ which piecewise continuous.

(a) If $|f(t)| \geq g(t)$ where $t \geq a$ for some positive constant M, and if $\int_M^\infty g(t)dt$ converges, then $\int_a^\infty f(t)dt$ converges.

(b) If $f(t) \geq g(t) \geq 0$ for $t \geq M$ and if $\int_M^\infty g(t)dt$ diverges, then $\int_a^\infty f(t)dt$ diverges.

Example: Suppose that (a) f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A. (b) $|f(t)| \leq Ke^{at}$ when $t \geq M$, where K, a, and M are real constants and K, M positive. Then, $\int_0^\infty e^{-st} f(t)dt$ exists for $s > a$.

The Laplace transform:

$$L\{f(t)\} \equiv F(s) = \int_0^\infty e^{st} f(t)dt$$

Examples

- $L\{1\} = \int_0^\infty e^{-st}dt = \frac{1}{-s}e^{-st}\Big|_0^\infty = -\frac{1}{s}[\underbrace{e^{-\infty}}_{=0} - e^0] = \frac{1}{s}$

- $L\{e^{at}\} = \int_0^\infty e^{at} \cdot e^{-st}dt = \frac{1}{a-s}e^{(a-s)t}\Big|_0^\infty = \begin{cases} \infty, & s < a \\ \frac{1}{s-a}, & s > A \end{cases}$

- $L\{\sin(at)\} = \int_0^\infty \sin(at)e^{-st}dt = \frac{a}{a^2+s^2} \quad (s > 0)$

  Properties: α, β numbers, $f(t)$, $g(t)$

$$L\{\alpha f(t) + \beta\beta(t)\} = \alpha L\{f(t)\} + \beta L\{g(t)\}.$$

4.2 Solution of Initial Value Problems

Theorem 4.2.1 Suppose that $|f(t)| \leq Ke^{at}$ for $t \geq M$. Then,

$$L\{f'(t)\} = sL\{f(t)\} - f(0). \quad (s > a)$$

Generalization:

$$L\{f^{(n)}(t)\} = S^n L\{f(t)\} - s^{n-1}f(0) - \cdots - f(n-1)(0)$$

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

Examples:

- $y'' - 2y' - 2y = 0, y(0) = 2, y'(0) = 0$

  Sol:

  $$y''(t) - 2y'(t) - 2y(t) = 0$$
  $$L\{y''(t) - 2y'(t) - 2y(t)\} = L\{0\} = \frac{0}{s} = 0$$
  $$L\{y''(t)\} - 2L\{y'(t)\} - 2L\{y(t)\} = 0$$

  Define: $Y(s) = L\{y(t)\}$

  $$L\{y'(t)\} = sY(s) - y(0) = sY(s) - 2$$
  $$L\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s$$
  $$[s^2 \cdot Y(s) - 2s] - 2[s \cdot Y(s) - 2] - 2Y(s) = 0$$
  $$[s^2 - 2s - 2]Y(s) = 2s - 4$$
  $$Y(s) = \frac{2s - 4}{s^2 - 2s - 2}$$

  The inverse Laplace transform:
  $$y(t) = L^{-1}\{Y(s)\}$$

- $y'' + 9y = \cos t$

4.3 Step functions

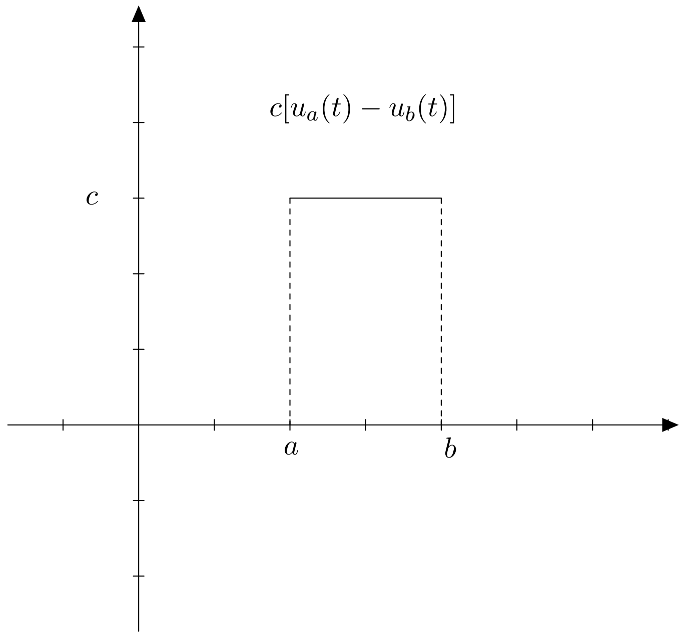
Definition of the unit step function:

$$u_c(t) = \begin{cases} 0, & \text{for } t \leq c \\ 1, & \text{for } t > c \end{cases}$$

or $1 - u_c(t)$

Hat function

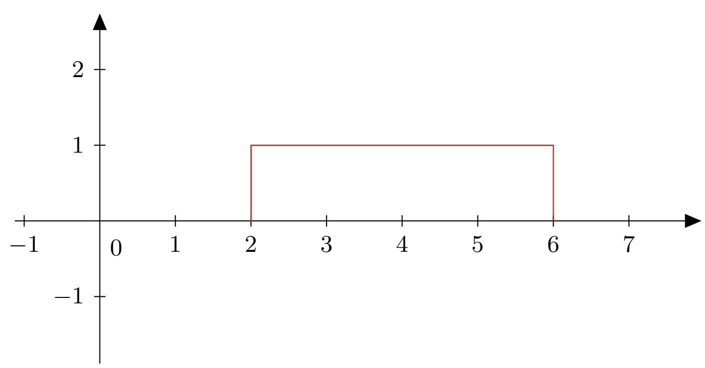
$$f(t) = \begin{cases} 0, & \text{for } t \leq a \\ c, & \text{for } a < t \leq b \\ 0, & \text{for } t > b \end{cases} = \boxed{c[u_a(t) - u_b(t)]}$$



Examples: Express the following functions in terms of unit step function.

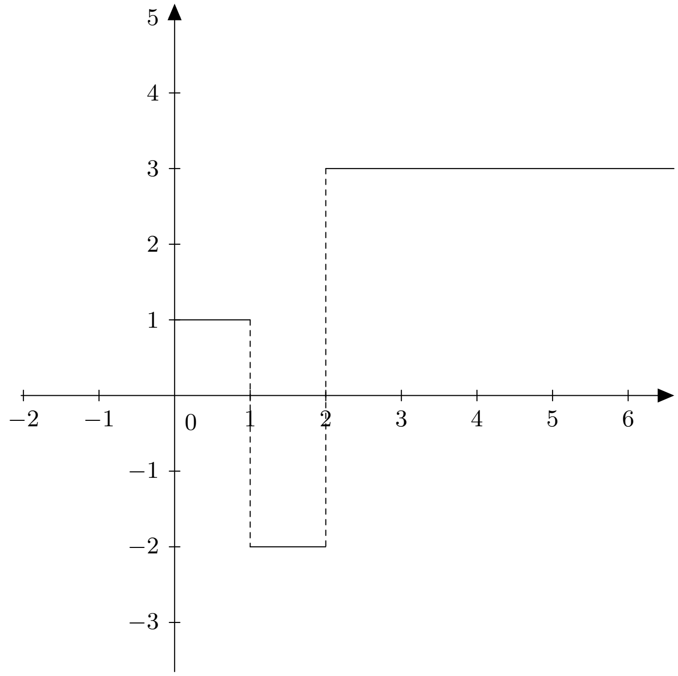
- $f(t) = \begin{cases} 0, & \text{for } t \le 2 \\ 1, & \text{for } 2 < t \le 6 \\ 0, & \text{for } t > 6 \end{cases}$

  Sol: $= u_2(t) - u_6(t)$.



- $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -2, & 1 \le t < 2 \\ 3, & t \ge 2 \end{cases}$

  sol: $f(t) = [u(t) - u_1(t)] + (-2) \cdot [u_1(t) - u_2(t)] + 3 \cdot u_2(t)$

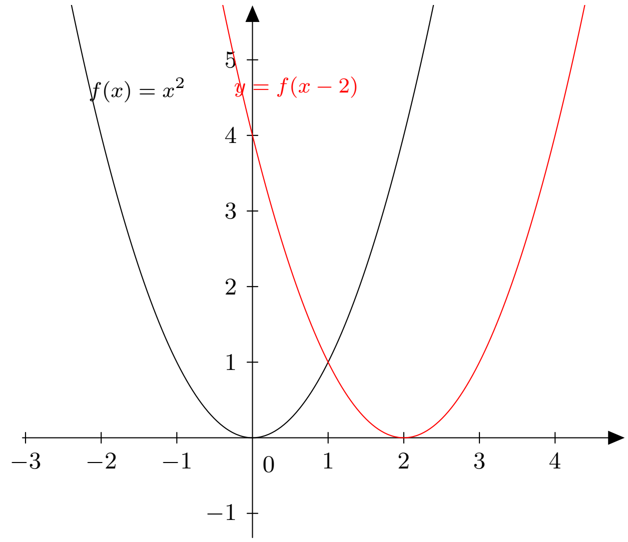


- $f(t) = \begin{cases} t, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ t^2, & t \ge 2 \end{cases}$

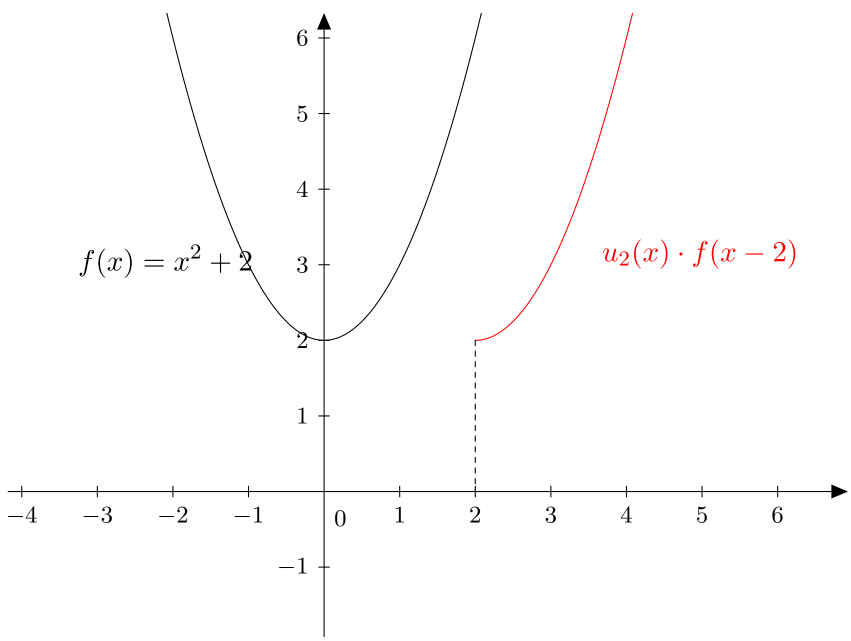
  Sol: $f(t) = t \cdot [u(t) - u_1(t)] + (t-1) \cdot [u_1(t) - u_2(t)] + t^2 \cdot u_3(t)$.

<u>Translation of a function</u>

- Given $b > 0$
  - $f(x - b)$ is the translation of $f(t)$ to the right by b units.
  - $f(x + b)$ is the translation of $f(t)$ to the left by b units.



- $u_c(t)f(t)$ is the cut-off of $f(t)$ at c.
- $u_c(t)f(t-b)$: move f by b units ((a)to the right, if $b > 0$; (b) to the left if $b < 0$). And then cut off $f(t-b)$ at c.



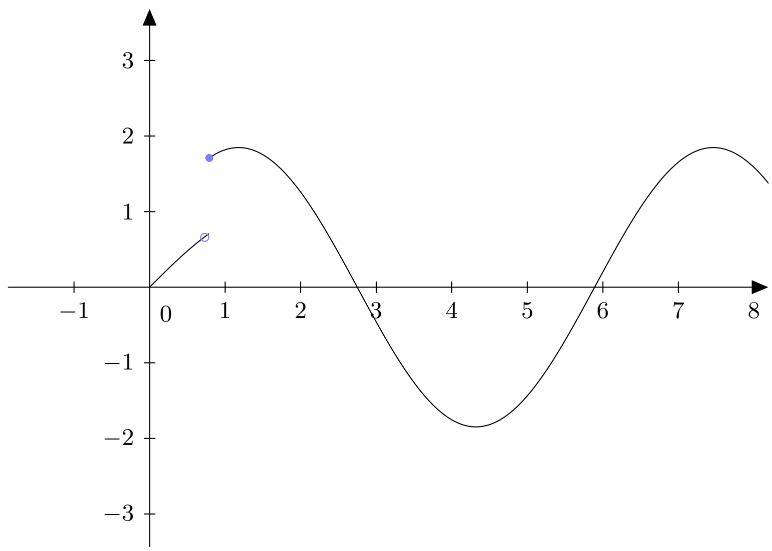
<u>**Laplace transforms**</u>: Suppose $F(s) = L\{f(t)\}$

- $L\{u_c(t)f(t-c)\} = e^{-cs}F(s)$
- $u_c(t)f(t-c) = L^{-1}\{e^{-cs}F(s)\}$

Example 1: Given

$$f(t) = \begin{cases} \sin t, & 0 \le t < \dfrac{\pi}{4} \\ \sin t + \cos(t - \pi/4), & t \ge \dfrac{\pi}{4} \end{cases}$$

find $L\{f(t)\}$.



Note that $f(t) = \sin t + g(t)$, where

$$g(t) = \begin{cases} 0, & t < \dfrac{\pi}{4} \\ \cos(t - \dfrac{\pi}{4}), & t \ge \dfrac{\pi}{4} \end{cases} = \overbrace{u_{\pi/4}}^{\text{cut off at } \pi/4} \cdot \underbrace{\cos(t - \pi/4)}_{\text{move } \cos t \text{ to the right by } \pi}$$

$$\begin{aligned} L\{f(t)\} &= L\{\sin t\} + L\{u_{\pi/4} \cos(t - \pi/4)\} \\ &= L\{\sin t\} + e^{-\frac{\pi s}{4}} L\{\cos t\} \\ &= \frac{1}{s^2 + 1} + e^{-\pi s/4} \frac{s}{s^2 + 1} \end{aligned}$$

Table of Laplace Transform

Integration Table

$f(t)$	$F(s) = L\{f(t)\}$
$af(t)+bg(t)$	$aF(s) + bF(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdot - f^{(n-1)}(0)$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$
$f(t+a)$	unkown
$u_af(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$u(t)$	$\frac{1}{s}$
$u(t-c)$	$\frac{e^{cs}}{s}$
$\frac{1}{t}$	$\frac{1}{s^2}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$, n is integer
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin(t)$	$\frac{1}{1+s^2}$
$\sin(at)$	$\frac{a}{a^2+s^2}$
$\cos(at)$	$\frac{s}{1+s^2}$
$\cos(at)$	$\frac{s}{a^2+s^2}$

Examples: Find the inverse Laplace transform of the given function.

- $G(s) = \frac{3!}{(s-2)^4}$.

  Soln: Let $F(s) = L\{t^3\} = \frac{3!}{s^4}$. Then $G(s) = F(s-2)$, $L^{-1}\{G(s)\} = e^{2t}t^3$.

- $G(s) = \frac{2s+2}{s^2+2s+5}$

  $$G(s) = \frac{2s+2}{s^2+2s+5} = \frac{2s+2}{(s+1)^2+4} = 2 \cdot \frac{s+1}{(s+1)^2+2^2}$$

  Let $F(s) = L\{\cos 2t\} = \frac{s}{s^2+2^2}$. Then $G(s) = F(s+1)$

  $L^{-1}\{G(s)\} = e^{-t}\cos(2t)$

- $G(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2}$

  Soln: $G(s) = 2e^{-2s}\dfrac{s-1}{(s-1)^2+1}$

  Let $F(s) = L\{\cos t\} = \dfrac{s}{s^2+1}$. Then $G(s) = 2e^{-2s}F(s-1)$

  $L^{-1}\{F(s-1)\} = e^t \cos t$

  $L^{-1}\{e^{-2s}F(s-1)\} = u_2(t)\cdot \underbrace{e^{t-2}\cos(t-2)}_{\text{replace } t \text{ by } t-2}$

  $L^{-1}\{2\cdot e^{-2s}F(s-1) = 2\cdot u_2(t)\cdot e^{t-2}\cos(t-2)$

- $G(s) = \dfrac{2e^{-3s}}{s^2-4}$

  $\dfrac{2e^{-3s}}{s^2-4} = 2e^{-3s}\cdot\dfrac{1}{(s-2)(s+2)} = 2e^{-3s}\cdot\left[\dfrac{1}{s-2}-\dfrac{1}{s+2}\right]\cdot\dfrac{1}{4} = \dfrac{1}{2}e^{-3s}\left[\dfrac{1}{s-2}-\dfrac{1}{s+2}\right]$

  Let $F(s) = L\{u(t)\} = \dfrac{1}{s}$, then $G(s) = \dfrac{1}{2}e^{-3t}[F(s-2)-F(s+2)]$

  $L^{-1}\{F(s-2)\} = e^{2t}u(t), \quad L^{-1}\{F(s+2)\} = e^{-2t}u(t)$

  $L^{-1}\{e^{-3s}F(s-2)\} = \underbrace{u_3(t)\cdot e^{2(t-3)}u(t-3)}_{\text{multiply } u_3(t)\text{, and replace } t \text{ by } t-3} = u_3(t)e^{2(t-3)}$

  $L^{-1}\{e^{-3s}F(s+2)\} = \underbrace{u_3(t)\cdot e^{-2(t-3)}u(t-3)}_{\text{multiply } u_3(t)\text{, and replace } t \text{ by } t-3} = u_3(t)e^{-2(t-3)}$

  $g(t) = \dfrac{1}{2}[u_3(t)e^{2(t-3)} - u_3(t)e^{-2(t-3)}]$

Chapter 6

Numerical Methods

Consider IVP

$$\frac{dy}{dt} = f(t, y),\ y(t_0) = y_0$$

Numerical methods

Given a set of discrete time instants

$$t_0 < t_1 < \cdots < t_n < \cdots < t_N$$

want to find approximations of the solution, i.e.,

$$\begin{aligned} y_0 &= y(t_0) \\ y_1 &\simeq y(t_1) \\ &\vdots \\ y_n &\simeq y(t_n) \\ &\vdots \\ y_N &\simeq y(t_N) \end{aligned}$$

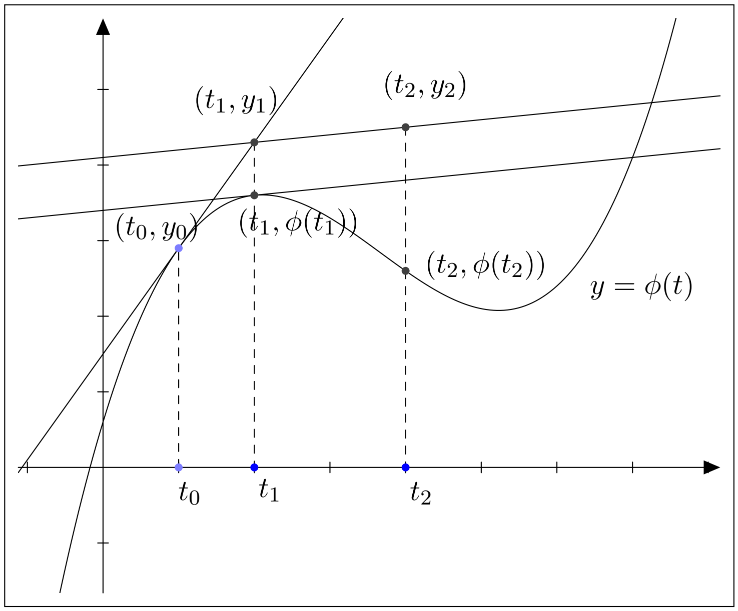
Often: for a positive time step $h > 0$

$$t_0,\ t_1 = t_0 + h, \cdots, t_{n+1} = t_n + h, \cdots$$

$$t_n = t_0 + nh,\ n = 0, 1, ..., N$$

6.1 Euler's method

Derivation



- Given $(t_0, y_0) \longrightarrow (t_1, y_1)$.
  tangent line: $y - y_0 = f(t_0, y_0)(t - t_0)$
  Set $t = t_1 \Rightarrow y_1 = y_0 + f(t_0, y_0)h$, $h = (t_1 - t_0)$. If h is small, (hope) $y_1 \simeq \phi(t_1)$.
- Given $(t_1, y_1) \longrightarrow (t_2, y_2)$.
  tangent line: $y - \phi(t_1) = f(t_1, \phi(t_1))(t - t_1)$
  approximation tangent line: $y - y_1 = f(t_1, y_1)(t - t_1)$
  $y_2 = y_1 + f(t_1, y_1)h$, h small $\Rightarrow y_2 \simeq \phi(t_2)$.
- Given $(t_n, y_n) \longrightarrow (t_{n+1}, y_{n+1})$
  $y_{n+1} = y_n + f(t_n, y_n)h$

Euler's method:

$$\boxed{y_{n+1} = y_n + f(t_n, y_n)h, \quad n = 0, 1, \cdots}$$

Error estimate:

Let $y = \phi(t)$ be the exact solution.

absolute error: $e_n = |y_n - \phi(t_n)|$

relative error: $\tilde{e}_n = \dfrac{|y_n - \phi(t_n)|}{|\phi(t_n)|}$

increment: $\Delta y_n = hf(t_n, y_n)$, $y_{n+1} = y_n + \Delta y_n$

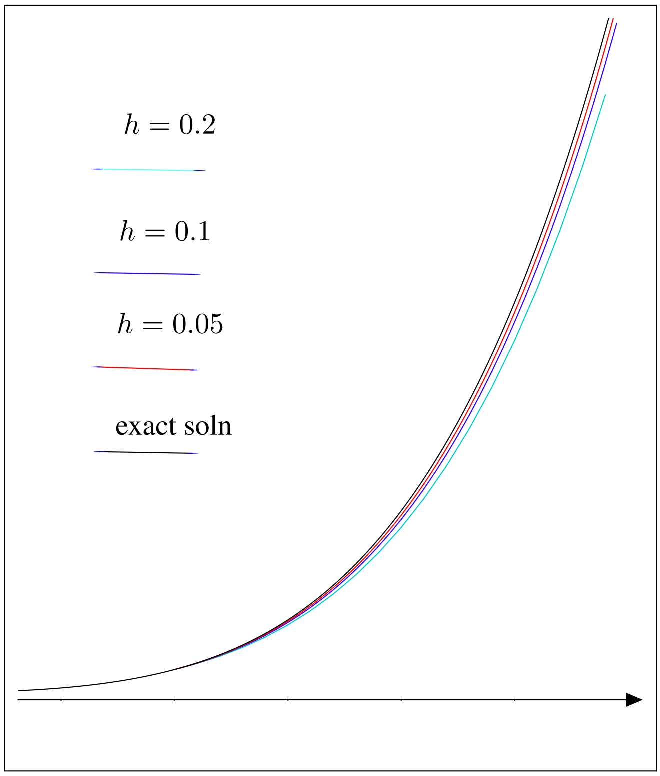
Example 1: Euler's method. $y' = t\sqrt{y}$, $y(2) = 4$. Use two steps with $h = 0.2$

n	t_n	y_n	$f(t_n, y_n)$	Δy_n	y_{n+1}
0	2	4	$2 \times \sqrt{4} = 4$	0.8	4.8
1	2.2	4.8	$2 \times \sqrt{4.8} = 4.382$	0.876	5.676
2	2.4	5.676			

Example 2: $y' = t\sqrt{y}, y(2) = 4$.

Exact solution: $y = \left(1 + \frac{t^2}{4}\right)^2$

(a) $h = 0.2$, 20 steps. (b) $h = 0.1$, 40 steps. (c)$h = 0.05$, 80 steps.



<u>Derivation II</u>: (Integration method)

$y = \phi(t)$ the exact solution of IVP $\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$

Thus, $\phi'(t) = f(t, \phi(t)), \Rightarrow \int_{t_n}^{t_{n+1}} \phi'(t)dt = \int_{t_n}^{t_{n+1}} f(t, \phi(t))dt$

So $\phi(t_{n+1}) - \phi(t_n) \simeq f(t_n, \phi(t_n))(t_{n+1} - t_n)$

or $\phi(t_{n+1}) \simeq \phi(t_n) + f(t_n, \phi(t_n))(t_{n+1} - t_n) \simeq y_n + f(t_n, y_n)h$

Backward Euler method:

$\phi(t_{n+1}) - \phi(t_n) \sim f(t_{n+1}, \phi(t_{n+1}))(t_{n+1} - t_n)$

$\phi(t_{n+1}) \sim \phi(t_n) + f(t_{n+1}, \phi(t_{n+1}))h \sim y_n + f(t_{n+1}, \phi(t_{n+1}))h$

Define: $y_{n+1} = y_n + f(t_{n+1}, y_{n+1})h$

Thus, the backward Euler method is also called the implicit Euler method (because the new time approximation is implicity defined).

Convergence analysis

One of the most important questions studied in Numerical Analysis.

Convergence: As the step size $h \to 0$, do the approximations of the solution y_1, y_2, ..., y_n approach the corresponding values of the exact solution? How fast?

Error (Global): $e_n = \phi(t_n) - y_n$.

Local truncation error: τ_n is the error in one step caused by the discretization (i.e. by assuming $y_n = \phi(t_n)$).

Local truncation error of Euler's method

$$y_{n+1} = y_n + f(t_n, y_n)h$$

Consider a time period: $t \in [0, T]$. Assume that $\phi = \phi(t)$ has continuous 2nd order derivative, i.e. $\phi''(t)$ continuous in $[0, T]$. (This can be shown by assuming f_t, f_y, f continuous.)

$$M_2 = \max_{0 \le t \le T} |\phi''(t)| \le \infty$$

Taylor Series:

$$\phi(t_{n+1}) = \phi(t_n) + \phi'(t_n)h + \frac{h^2}{2}\phi''(\bar{t})$$

$$\phi(t_{n+1}) = \phi(t_n) + f(t_n, \phi(t_n))h + \frac{h^2}{2}\phi''(\bar{t})$$

$$\phi(t_{n+1}) - y_{n+1} = \phi(t_n) - y_n + [f(t_n, \phi(t_n)) - f(t_n, y_n)]h + \frac{h^2}{2}\phi''(\bar{t})$$

$$e_{n+1} = e_n + [f(t_n, \phi(t_n)) - f(t_n, y_n)]h + \frac{h^2}{2}\phi''(\bar{t})$$

Assuming $y_n = \phi(t_n)$ (i.e. $e_n = 0$), then

$$\boxed{\tau_{n+1} = e_{n+1} = \frac{h^2}{2}\phi(\bar{t})}$$

$$\therefore |\tau_{n+1}| \le \frac{h^2}{2}M_2$$

It can be proven that

$$|e_{n+1}| \leq C_1 N |\tau_{n+1}| \leq Ch$$

Euler's method is a *first order method* method = the order of $|\tau_n| - 1$.

Exercise Backward Euler: $y_{n+1} = y_n + f(t_{n+1}, y_{n+1})h$

$\phi(t_n) = \phi(t_{n+1}) - \phi'(t_{n+1})h + \frac{h^2}{2}\phi''(\bar{t})$

or

$$\begin{aligned}\phi(t_{n+1}) &= \phi(t_n) + \phi'(t_{n+1})h - \frac{h^2}{2}\phi''(\bar{t}) \\ &= \phi(t_n) + f(t_{n+1}, \phi(t_{n+1}))h - \frac{h^2}{2!}\phi''(\bar{t})\end{aligned}$$

$e_{n+1} = e_n + [f(t_{n+1}, \phi(t_{n+1})) - f(t_{n+1}, y_{n+1})]h - \frac{h^2}{2}\phi''(\bar{t})$

$|e_{n+1}| \leq |e_n| + |f_y(t_{n+1}, \bar{y})| \cdot |e_{n+1}|h + \frac{h^2}{2}M_2$

Let $M_{1y} := \max_{t,y} |f_y(t, y)|$.

$(1 - M_{1y}h) \cdot |e_{n+1}| \leq |e_n| + \frac{h^2}{2!}M_2$

If $e_n = 0$ and h is small enough, such that $M_{1y}h \leq \frac{1}{2}$, then

$$\boxed{|\tau_{n+1}| = |e_{n+1}| \leq h^2 M_2}$$

6.2 8.2 Improvements on Euler's Method

Integration method

$\int_{t_n}^{t_{n+1}} \phi'(t)dt = \int_{t_n}^{t_{n+1}} f(t, \phi(t))dt$

$$\begin{aligned}\phi(t_{n+1}) &\simeq \phi(t_n) + \frac{1}{2}(t_{n+1} - t_n)[f(t_n, \phi(t_n)) + f(t_{n+1}, \phi(t_{n+1}))] \\ &\simeq y_n + \frac{h}{2}[f(t_n, \phi(t_n)) + f(t_{n+1}, \phi(t_{n+1}))]\end{aligned}$$

One way: define implicitly y_{n+1} through:

$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$

Another: $\phi(t_{n+1}) \simeq y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + f(t_n, y_n)h)]$

Define

$$\boxed{y_{n+1} \simeq y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + f(t_n, y_n)h)]}$$

This is called ”the improved Euler method” or ”Hewn’s method”.

Convergence properties

$|\tau_{n+1}| \leq Ch^3$

$|e_{n+1}| \leq Ch^2$ (second order method: if a step size is reduced by a factor of 2, error is reduced by a factor of 4).

formula for computation by hand: Let

- $\overline{\Delta y}_n = f(t_n, y_n)h$
- $\bar{y}_{n+1} = y_n + \bar{\Delta} y_n$
- $\overline{\overline{\Delta y}}_n = f(t_{n+1}, \bar{y}_{n+1})h$
- $y_{n+1} = y_n + \frac{1}{2}(\overline{\Delta y}_n + \overline{\overline{\Delta y}}_n)$

Example 1 Comparison of errors using Euler’s method and Heun’s method.

$$y' = 1 - t + 4y, \quad y(0) = 1$$

	Euler’s		Heun’s	
t	$h = 0.01$	$h = 0.001$	$h = 0.025$	$h = 0.01$
0.1	1.38e-2	1.41e-3	1.10e-3	1.83e-4
0.5	3.35e-1	3.49e-2	2.71e-2	4.54e-3

Example 2 $y' = 2y - 3t$, $y(0) = 1$, $h = 0.05$, find approximate value at $t = 0.1$ using Heun’s method.

n	t_n	y_n	$\overline{\Delta y}_n$	$\bar{y}_{n+1}$	$\overline{\overline{\Delta y}}_n$	Δy_n	y_{n+1}
0	0	1	0.1	1.1	0.103	0.102	1.102
1	0.05	1.102	0.103	1.205	0.106	0.104	1.206
2	0.1	1.206					