Homework Problems

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HW 1 (due June 8, Wednesday) Consider the BTCS scheme applied to the IBVP

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x}, \quad x \in (0,1) \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u^0(x) \end{cases}$$

on a uniform mesh of $J+1$ points, where b is a positive constant.

1. Define and calculate the local truncation error. Is the scheme consistent with the PDE under consideration?
2. Give the Fourier stability analysis for the scheme.
3. Give the L^∞ stability analysis for the scheme. Discuss the difference (if any) between the stability conditions obtained using the two methods and how these conditions can be satisfied.
4. Give the L^∞ convergence analysis for the scheme.

HW 2 (due June 15, Wednesday) Consider the Crank-Nicolson scheme applied to

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, & \text{in } \Omega = (0,1)\times(0,1) \\ u = 0, & \text{on } \partial\Omega \\ u(x,y,0) = u^0(x,y) \end{cases}$$

on a Cartesian grid. Repeat the four tasks listed in HW 1 for this scheme.

HW 3 (due June 22, Wednesday) Consider the TPBVP

$$\begin{cases} -(au')' = f, \quad \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

where $a = a(x)$ and $f = f(x)$ are given functions and

$$a(x) \geq \alpha > 0, \quad \forall x \in (0,1).$$

We assume that a uniform mesh with $J+1$ points is used.

1. Derive the Galerkin formulation for the TPBVP.
2. Describe the linear finite element solution of the TPBVP.

3. Give the convergence analysis for the linear FEM in the energy norm. You can use the linear interpolation error estimates discussed in the class.
4. Give the L^2 error analysis using the Aubin-Nischke trick (the duality argument).

HW 4 (due June 29, Wednesday) Given a mesh

$$x_0 = a < x_1 < \cdots < x_J = b,$$

the composite trapezoidal rule and the error are given by

$$\int_a^b f(x)dx = \sum_{j=1}^{J} \frac{h_j}{2} \left(f(x_{j-1}) + f(x_j) \right) - \frac{1}{J} \sum_{j=1}^{J} h_j^3 f''(\bar{x}_j),$$

where $\bar{x}_j$ is a point in (x_{j-1}, x_j).

1. Define the equidistributing mesh and the monitor function based on the error formulation.
2. What is the optimal upper bound of the error for the equidistributing mesh?
3. How to choose the regularization parameter for this problem?
4. What is the continuous form of the regularized monitor function and the equidistribution principle?