Lecture 11

Variational formulation of a TPBVP

- function spaces
- Examples of function spaces
- variational formulation

Function spaces

- Linear function spaces
  V is a collection of functions defined on $[0,1]$ and satisfies:
  $$\alpha u + \beta v \in V, \quad \forall u, v \in V, \ \alpha, \beta \in \mathbb{R}$$
- normed function spaces
  a norm is a nonnegative function defined on V:
  $$\|\cdot\| : V \to \mathbb{R}^+.$$
  and satisfying
  (i) positivity $\quad \|v\| \geq 0 \quad \forall v \in V$
  (ii) $\quad \|v\| = 0 \Rightarrow v = 0$
  (iii) ~~linearity~~ $\quad \|\alpha v\| = |\alpha| \cdot \|v\| \quad \forall \alpha \in \mathbb{R}, v \in V$
  (iv) triangle inequality
  $$\|u + v\| \leq \|u\| + \|v\|$$

– V ⊕ norm = normed space

– a ~~positive~~ nonnegative function satisfying (i) (iii) and (iv) is called a <u>semi-norm</u>.

— With norm we can measure the "distance" between two functions. $\|u-v\|$

— With norm we can consider convergence

— Convergent sequences:

$\{v_n\} \subset V,\ v \in V$:

$$\|v_n - v\| \to 0 \quad \text{as } n \to \infty.$$

- Banach spaces = normed linear function spaces ⊕ any cauchy seq. is a convergent seq.

Cauchy sequences: $\{v_n\}$: $\|v_n - v_m\| \to 0$ as $n, m \to \infty$

- Hilbert space = Banach space ⊕ inner product

Examples of function spaces

- Continuous functions

$$C[0,1] = \{ v \mid v \text{ is continuous on } [0,1] \}$$

$$\|v\|_{C[0,1]} = \max_{0 \le x \le 1} |v(x)|$$

Banach space

$$C^m[0,1] = \{ v \mid v^{(k)} \text{ is continuous on } [0,1],\ k = 0, 1, \dots, m \}$$

$$\|v\|_{C^m[0,1]} = \max_{k=0,\dots,m} \max_{0 \le x \le 1} |v^{(k)}(x)|$$

$$C[0,1] \supset C^1[0,1] \supset \cdots \supset C^\infty[0,1]$$

- Lebsque spaces $L^p(0,1)$

$$L^2(0,1) = \{ v \mid \int_0^1 v^2 dx < +\infty \}$$

$$\|v\|_{L^2(0,1)} = \left[\int_0^1 v^2 dx \right]^{1/2}$$

Hilbert space

inner product $(u,v) = \int_0^1 u v \, dx$

$$\|u\|^2 = (u,v)$$

$$L^1(0,1) = \{ v \mid \int_0^1 |v| \, dx < +\infty \}$$

$$\|v\|_{L^1(0,1)} = \int_0^1 |v| dx$$

Banach space

$$L^p(0,1) = \{ v \mid \int_0^1 |v|^p dx < +\infty \}$$

$$\|v\|_{L^p(0,1)} = \left[\int_0^1 |v|^p dx \right]^{\frac{1}{p}}$$

Banach space

$1 \le p \le +\infty$

$$L^\infty(0,1): \quad \|v\|_{L^\infty(0,1)} = \lim_{p \to \infty} \|v\|_{L^p(0,1)} = \sup_{0<x<1} |v(x)|$$

$$L^1(0,1) \supset L^2(0,1) \supset \cdots \supset L^\infty(0,1)$$

$$v(x) = \frac{1}{x^{1/2}} \in L^1(0,1)$$

$$\notin L^2(0,1)$$

- Sobolev spaces $H^m(0,1)$

$$H^1(0,1) = \{ v \mid v \in L^2(0,1),\ v' \in L^2(0,1) \}$$

$$\|v\|_{H^1(0,1)} = \left(\|v\|^2_{L^2(0,1)} + \|v'\|^2_{L^2(0,1)} \right)^{1/2}$$
$$= \left[\int_0^1 |v|^2 + |v'|^2 \, dx \right]^{1/2}$$

$$|v|_{H^1(0,1)} = \left(\int_0^1 |v'|^2 dx \right)^{1/2}$$
(semi-norm)

$$H^2(0,1) = \{ v \mid v \in L^2(0,1), v' \in L^2, v'' \in L^2 \}.$$

$H^m(0,1)$: $m = 1, 2, \cdots$

$$H^0(0,1) = L^2(0,1) \supset H^1(0,1) \supset H^2(0,1) \supset \cdots$$

Variational formulation

TPBVP (two-point)

$$(D) \begin{cases} -u'' = f & x \in (0,1) \\ u(0) = 0, \ u(1) = 0 \end{cases}$$

Define $H^1_0(0,1) = H^1(0,1) \cap \{ v(0) = v(1) = 0 \}$
$$= \{ v \mid v \in H^1(0,1), v(0) = v(1) = 0 \}.$$

Multiplying PDE with $v \in H_0^1(0,1)$
and integrating over $(0,1)$:

$$-\int_0^1 u'' v \, dx = \int_0^1 f v \, dx$$

$$\int_0^1 v' u' \, dx = \int_0^1 f v \, dx$$

Galerkin/Variational/weak formulation:

Find $u \in H_0^1(0,1)$ such that

$$(V) \qquad \int_0^1 u' v' dx = \int_0^1 f v dx \quad \forall v \in H_0^1(\Omega)$$

Minimization formulation:

Find $u \in H_0^1(0,1)$ such that

$$F(u) \leq F(v) \quad \forall v \in H_0^1(0,1)$$

where $F(v) = \frac{1}{2}(v', v') - (f, v)$

$$= \int_0^1 \left(\frac{1}{2}|v'|^2 - f v\right) dx$$

Relation

$$(D) \Rightarrow (V) \Longleftrightarrow (M)$$

(V) has a unique solution

#

Lecture 12

FEM for TPBVP

- partition the domain
- define FEM space
- define FEM approximation
- Implementation

$$\begin{cases} -u'' = f & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Galerkin formulation

$$\int_0^1 u'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in H_0^1(0,1)$$

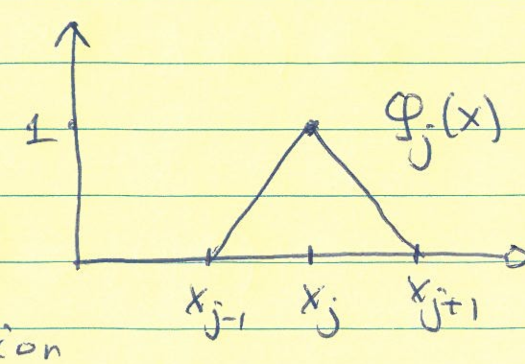
$$(u', v') = (f, v) \quad \forall v \in H_0^1(0,1)$$

▶ partition of the domain
(mesh, grid, triangulation)

$$x_0 \equiv 0 < x_1 < \cdots < x_J = 1$$

$$h_j = x_j - x_{j-1}$$

$$h = \max_j h_j$$



▶ FEM space

Define linear basis function
(tent function)

$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}} & x \in [x_{j-1}, x_j] \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j} & x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise} \end{cases}$$

$\varphi_j(x)$: piecewise linear, $C^0[0,1] = C[0,1]$.

$$S_h = \{ v_h \mid v_h = \sum_{j=1}^{J-1} \eta_j \varphi_j(x),\ \eta_j \in \mathbb{R} \}$$

$$= \text{span}\{\varphi_1, \cdots, \varphi_{J-1}\} \subset H_0^1(0,1)$$

- FEM approximation (V_h):

  Find $u_h \in S_h$ such that
  $$(u_h', v_h') = (f, v_h) \quad \forall v_h \in S_h$$

~~Implementation~~

Matrix form:

$$u_h = \sum_{j=1}^{J-1} u_j \varphi_j(x)$$

$$u_j \approx u(x_j)$$

$$(u_h', v_h') = (f, v_h)$$

$$\Longrightarrow \sum_{j=1}^{J-1} u_j (\varphi_j', v_h') = (f, v_h) \quad \forall v_h \in S_h$$

Take $v_h = \varphi_k(x)$ $(k = 1, \cdots, J-1)$:

$$\sum_{j=1}^{J-1} u_j \underbrace{(\varphi_j', \varphi_k')}_{a_{kj}} = \underbrace{(f, \varphi_k)}_{b_k} \quad k = 1, \cdots, J-1$$

$$\sum_{j=1}^{J-1} a_{kj} u_j = b_k \quad k = 1, \cdots, J-1$$

$$A\vec{u} = \vec{b}$$

$$A = (a_{kj})_{(J-1)\times(J-1)}$$

$$a_{kj} = (\varphi_j', \varphi_k') = \int_0^1 \varphi_j' \varphi_k' \, dx$$

$$b_k = (f, \varphi_k) = \int_0^1 f \varphi_k \, dx$$

Properties A — stiffness matrix

(i) A : symmetric and positive definite

(ii) $$a_{kj} = \int_{x_{j-1}}^{x_j} \varphi_j' \varphi_k' \, dx + \int_{x_j}^{x_{j+1}} \varphi_j' \varphi_k' \, dx$$

$$= \frac{1}{h_j} \int_{x_{j-1}}^{x_j} \varphi_k' \, dx - \frac{1}{h_{j+1}} \int_{x_j}^{x_{j+1}} \varphi_k' \, dx$$

$$= \frac{1}{h_j}\left(\varphi_k(x_j) - \varphi_k(x_{j-1})\right)$$

$$- \frac{1}{h_{j+1}}\left(\varphi_k(x_{j+1}) - \varphi_k(x_j)\right)$$

$$\left.\begin{aligned} a_{k,k} &= \frac{1}{h_k} + \frac{1}{h_{k+1}} \\ a_{k,k-1} &= -\frac{1}{h_k} \\ a_{k,k+1} &= -\frac{1}{h_{k+1}} \end{aligned}\right\}$$ A is diagonally dominant

$$A = \begin{bmatrix} \ddots & \ddots & \\ \ddots & \ddots & \ddots \\ & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} \frac{1}{h_1} + \frac{1}{h_2} & -\frac{1}{h_2} & & \\ -\frac{1}{h_2} & \frac{1}{h_2} + \frac{1}{h_3} & -\frac{1}{h_3} & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

(iii) $$b_k = \int_{x_{k-1}}^{x_k} f(x) \varphi_k(x) \, dx + \int_{x_k}^{x_{k+1}} f(x) \varphi_k(x) \, dx$$

Numerical integration — quadrature

For example:

2-point Gaussian quadrature rule

$$\int_{-1}^{1} f(x)\,dx \approx f\left(\tfrac{1}{\sqrt{3}}\right) + f\left(-\tfrac{1}{\sqrt{3}}\right)$$

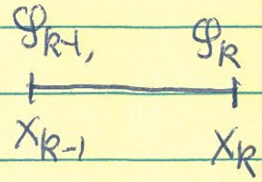
$$\int_{x_{k-1}}^{x_k} f(x)\varphi_k(x)\,dx \approx h_k f\left(x_{k-1} + \frac{1+\frac{1}{\sqrt{3}}}{2} h_k\right)\varphi_k(\cdots) + h_k f\left(x_{k-1} + \frac{1-\frac{1}{\sqrt{3}}}{2} h_k\right)\varphi_k(\cdots)$$

Implementation (elementwise assembling)

- Set $A := 0$, $b := 0$ (initialization)
- For $k = 1, \cdots, J$ (on element (x_{k-1}, x_k))

$$\begin{cases} a_{kk} := a_{kk} + \int_{x_{k-1}}^{x_k} \varphi_k' \varphi_k'\,dx \\ a_{k-1,k-1} := a_{k-1,k-1} + \int_{x_{k-1}}^{x_k} \varphi_{k-1}' \varphi_{k-1}'\,dx \\ a_{k,k-1} := a_{k,k-1} + \int_{x_{k-1}}^{x_k} \varphi_{k-1}' \varphi_k'\,dx \\ a_{k-1,k} := a_{k-1,k} + \int_{x_{k-1}}^{x_k} \varphi_k' \varphi_{k-1}'\,dx \end{cases}$$

$$\begin{cases} b_k := b_k + \int_{x_{k-1}}^{x_k} f(x)\varphi_k\,dx \\ b_{k-1} := b_{k-1} + \int_{x_{k-1}}^{x_k} f(x)\varphi_k\,dx \end{cases}$$



Local stiffness matrix $\begin{bmatrix} (\varphi_{k-1}', \varphi_{k-1}')_k & (\varphi_k', \varphi_{k-1}')_k \\ (\varphi_{k-1}', \varphi_k')_k & (\varphi_k', \varphi_k')_k \end{bmatrix}$

Local rhs

$$\begin{bmatrix} (f, \varphi_{k-1})_k \\ (f, \varphi_k)_k \end{bmatrix}$$

#

Lecture 13

Error estimates for linear FiEM

- error equation
- error estimate in the energy norm
- error estimate in the L^2 norm
- interpolation error

The error equation

Recall:

Continuous proble: find $u \in H_0^1(0,1)$

$$(u', v') = (f, v) \quad \forall v \in H_0^1(0,1)$$

FiEM approximation: find $u_h \in S_h \subset H_0^1(0,1)$

$$(u_h', v_h') = (f, v_h) \quad \forall v_h \in S_h$$

$\Rightarrow$ orthogonality: $e_h = u - u_h$

$$((u-u_h)', v_h') = 0 \quad \forall v_h \in S_h$$

or

$$(e_h', v_h') = 0 \quad \forall v_h \in S_h$$

or

$e_h \perp S_h$ in the energy norm $|v|_{H^1(0,1)}$

~~error estimate in the energy norm~~

$\Rightarrow$ the error equation: for any $v \in H_0^1(0,1)$

$$(e_h', v') = (e_h', v' - v_h') + (e_h', v_h')$$

$$(e_h', v') = (e_h', v' - v_h') \quad \forall v \in H_0^1(\Omega), \; v_h \in S_h$$

The error estimate in the energy norm

The error eqn:

$$(e_h', v') = (e_h', v' - v_h')$$

Take $v = e_h$

$$\Rightarrow \quad (e_h', e_h') = (e_h', e_h' - v_h') = (e_h', u' - v_h')$$

Schwartz inequality:

$$\left|\sum_{i=1}^{n} a_i b_i\right| \le \left(\sum_i a_i^2\right)^{\frac{1}{2}} \left(\sum_i b_i^2\right)^{1/2}$$

$$\left|\int_0^1 u v \, dx\right| \le \left(\int_0^1 u^2 dx\right)^{1/2} \left(\int_0^1 v^2 dx\right)^{1/2}$$

$$|(u, v)| \le \|u\|_{L^2(0,1)} \|v\|_{L^2(0,1)}$$

$$\Rightarrow \quad \|e_h'\|^2_{L^2(0,1)} \le \|e_h'\|_{L^2(0,1)} \|u' - v_h'\|_{L^2(\Omega)}$$

or

$$\|e_h'\|_{L^2(0,1)} \le \|u' - v_h'\|_{L^2(\Omega)}$$

Since v_h is arbitrary, we have

$$\boxed{\|e_h'\|_{L^2(0,1)} \le \inf_{v_h \in S_h} \|u' - v_h'\|_{L^2(0,1)}}$$

☆ The energy norm of the solution error is bounded by the best approximation error in the energy norm

☆ We choose v_h to be the linear interpolant of u:

$$v_h = I_h u \equiv \sum_{j=0}^{J} u(x_j)\varphi_j(x)$$

Then $$\|e_h'\|_{L^2(0,1)} \leq \|u' - (I_h u)'\|_{L^2(0,1)}$$

<u>Proposition</u> (Interpolation error)

$$|u'(x) - (I_h u)'(x)| \leq h_j \max_{x_{j-1} \leq y \leq x_j} |u''(y)|$$

$$|u(x) - (I_h u)(x)| \leq \frac{h_j^2}{8} \max_{x_{j-1} \leq y \leq x_j} |u''(y)|$$

$$\forall\, x \in [x_{j-1}, x_j)$$

Thus,

$$\begin{aligned}
\|e_h'\|^2_{L^2(0,1)} &\leq \|u' - (I_h u)'\|^2_{L^2(0,1)} \\
&= \sum_{j=1}^{J} \int_{x_{j-1}}^{x_j} |u' - (I_h u)'|^2\, dx \\
&\leq \sum_{j=1}^{J} h_j^3 \max_{x_{j-1} \leq y \leq x_j} |u''(y)|^2 \\
&\leq \left(\sum_{j=1}^{J} h_j^3\right) \max_{0 \leq y \leq 1} |u''(y)|^2 \\
&\leq h^2 \max_{0 \leq y \leq 1} |u''(y)|^2
\end{aligned}$$

or:

$$\boxed{\|e_h'\|_{L^2(0,1)} \leq h \max_{0 \leq y \leq 1} |u''(y)|} = O(h)$$

Error estimates in the L^2 norm

Poincare's inequality

$$\|v'\|_{L^2(0,1)} \geq C_p \|v\|_{L^2(0,1)} \quad \forall v \in H_0^1(0,1)$$

$$\Rightarrow \quad \|e_h\|_{L^2(0,1)} \leq \frac{1}{C_p} h \max_{0\leq y\leq 1} |u''(y)|$$
$$= O(h)$$

But interpolation error in $L^2(0,1)$ is $O(h^2)$.

Aubin-Nitsche trick: (duality argument)

Define the dual problem

$$\varphi'' = e_h \quad \varphi(0) = \varphi(1) = 0$$

- Known: regularity of this continuous problem
$$\|\varphi\|_{H^2(0,1)} \leq C \|e_h\|_{L^2(0,1)}$$
- $(e_h, e_h) = (e_h, \varphi'')$
$$= -(e_h', \varphi') \quad \text{(integration by parts)}$$
$$= -(e_h', (\varphi - I_h\varphi)') \quad \text{(orthogonality)}$$

$$\Rightarrow \|e_h\|^2_{L^2(0,1)} \leq \|e_h'\|_{L^2(0,1)} \|\varphi' - (I_h\varphi)'\|_{L^2(0,1)}$$
$$\leq C h \max_y |u''(y)| \cdot h \|\varphi''\|_{L^2(0,1)}$$
$$\leq C h^2 \|e_h\|_{L^2(0,1)} \max_y |u''(y)|$$

$$\Rightarrow \boxed{\|e_h\|^2_{L^2(0,1)} \leq C h^2 \max_y |u''(y)|} \quad \#$$

Interpolation error

estimate $(u - I_h u)(x)$ for $x \in (x_{j-1}, x_j)$

$$I_h u(x) = \frac{x - x_{j-1}}{x_j - x_{j-1}} u(x_j) + \frac{x_j - x}{h_j} u(x_{j-1})$$

Taylor expansion

$$u(x_j) = u(x_j - x + x)$$
$$= u(x) + (x_j - x) u'(x) + \frac{(x_j - x)^2}{2} u''(\xi)$$

$$u(x_{j-1}) = u(-x + x_{j-1} + x)$$
$$= u(x) + (x_{j-1} - x) u'(x) + \frac{(x_{j-1} - x)^2}{2} u''(\eta)$$

$$u(x) - I_h u(x) = - \frac{(x - x_{j-1})(x_j - x)^2}{2 h_j} u''(\xi)$$
$$- \frac{(x_j - x)(x_{j-1} - x)^2}{2 h_j} u''(\eta)$$

$$\Rightarrow \quad |u(x) - I_h u(x)| \leq \frac{h_j^2}{8} \max_{x_{j-1} \leq y \leq x_j} |u''(y)|$$

$$(I_h u)'(x) = \frac{1}{h_j} \left(u(x_j) - u(x_{j-1}) \right)$$

$$\Rightarrow \quad |u'(x) - (I_h u)'(x)| \leq h_j \max_{x_{j-1} \leq y \leq x_j} |u''(y)| .$$

#

Lecture 14

FEM for the Poisson equation

- Galerkin formulation
- Mesh terminology
- Linear finite element space
- Linear FEM and stiffness matrix
- Error analysis
- Examples of meshes

Galerkin/Weak formulation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Ω is assumed to be polygonal.

$H_0^1(\Omega) = \{ v \mid v \in H^1(\Omega),\ v|_{\partial\Omega} = 0 \}$

For any $v \in H_0^1(\Omega)$:

$$-\int_\Omega \Delta u \, v \, dx\,dy = \int_\Omega f v \, dx\,dy$$

$$v \Delta u = \nabla \cdot (v \nabla u) - \nabla v \cdot \nabla u$$

$$\Rightarrow \quad -\int_\Omega \nabla \cdot (v \nabla u)\, dx\,dy + \int_\Omega \nabla u \cdot \nabla v \, dx\,dy = \int_\Omega f v \, dx\,dy$$

$\Downarrow$ divergence Thm (Gauss' Thm)

$$-\underbrace{\int_{\partial\Omega} v \nabla u \cdot \vec{n}\, dl}_{0} + \int_\Omega \nabla u \cdot \nabla v \, dx\,dy = \int_\Omega f v \, dx\,dy$$

Galerkin formulation: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega)$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx\,dy$$

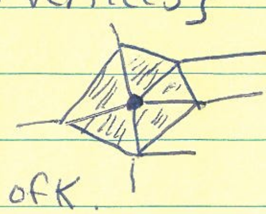
$$(f,v) = \int_\Omega f v \, dx\,dy$$

Mesh terminology

Ω : polygonal domain (to void boundary approximation)

$\mathcal{T}_h = \{K_1, \cdots, K_m\}$: mesh/triangulation for Ω

- K_i, $i=1,\cdots,m$ triangles
- Non-overlapping: $(K_i \setminus \partial K_i) \cap (K_j \setminus \partial K_j) = \phi$, $i \neq j$
- $\bigcup_{i=1}^{m} K_i = \Omega$
- No hanging point: no vertex of a triangle lies on the edges of another triangle
- interior nodes/vertices: N_j, $j=1,\cdots,J$ $(N_j \notin \partial\Omega)$
- element patch $\omega_j = \{K :$ all elements having N_j as one of its vertices$\}$
- h_K (diameter of K): length of the longest edge of K.



ω_j

- $h = \max_{K \in \mathcal{T}_h} h_K$

Linear finite element space S_h

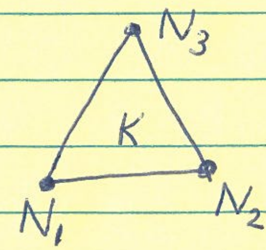
$$S_h = \left\{ v_h : \begin{array}{l} v_h \in C^0(\Omega) \\ v_h|_K \text{ linear} \quad \forall K \in \mathcal{T}_h \\ v_h|_{\partial\Omega} = 0 \end{array} \right\}$$

$S_h \subset H_0^1(\Omega)$

$S_h = \text{span}\{\varphi_1, \cdots, \varphi_J\}$

linear basis function φ_j (associated with node N_j):

- $\varphi_j \in C^0(\Omega)$
- $\varphi_j|_K$ is linear
- $\varphi_j(N_k) = \delta_{jk} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$
- $\sup \varphi_j = \omega_j$
- $\varphi_1(x,y)$



$$= \frac{\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}$$

Linear FEM and Stiffness matrix

- Find $u_h \in S_h^1$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in S_h^1$$

- Matrix form:

$$u_h = \sum_{j=1}^{J} u_j \varphi_j(x,y)$$

$$\Rightarrow \quad \sum_{j=1}^{J} u_j \, a(\varphi_j, v_h) = (f, v_h) \quad \forall v_h \in S_h^1$$

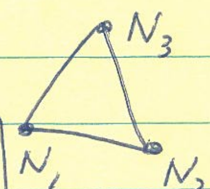
Taking $v_h = \varphi_k$

$$\Rightarrow \quad \sum_{j=1}^{J} a(\varphi_j, \varphi_k) u_j = (f, \varphi_k) \quad k = 1, \cdots, J$$

$$\Rightarrow \quad A\vec{u} = \vec{b}$$

$$a_{kj} = a(\varphi_j, \varphi_k), \quad b_k = (f, \varphi_k)$$

- Local Stiffness matrix and RHS



$$\begin{bmatrix} a_K(\varphi_1,\varphi_1) & a_K(\varphi_2,\varphi_1) & a_K(\varphi_3,\varphi_1) \\ \bullet & a_K(\varphi_2,\varphi_2) & a_K(\varphi_3,\varphi_2) \\ & & a_K(\varphi_3,\varphi_3) \end{bmatrix} \quad \begin{bmatrix} (f,\varphi_1)_K \\ (f,\varphi_2)_K \\ (f,\varphi_3)_K \end{bmatrix}$$

A symmetric and positive definite

Error analysis

Recall
$$a(u,v)=(f,v) \quad \forall v\in H_0^1(\Omega)$$
$$a(u_h,v_h)=(f,v_h) \quad \forall v_h\in S_h$$

$\Rightarrow$ orthogonality:
$$a(u-u_h,v_h)=0 \quad \forall v_h\in S_h$$

$e_h=u-u_h$: $\boxed{a(e_h,v_h)=0 \quad \forall v_h\in S_h}$

$$\begin{aligned}\|\nabla e_h\|^2_{L^2(\Omega)} &= a(e_h,e_h)\\ &= a(e_h, u-v_h+v_h-u_h)\\ &= a(e_h,u-v_h) \quad \text{(orthogonality)}\\ &\leq \|\nabla e_h\|_{L^2(\Omega)}\|\nabla(u-v_h)\|_{L^2(\Omega)}\\ &\qquad \text{(Schwartz inequality)}\end{aligned}$$

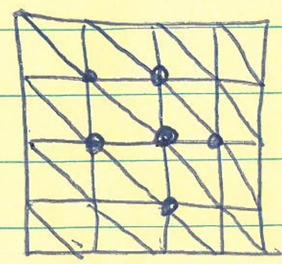
$\Rightarrow$
$$\boxed{\|\nabla e_h\|_{L^2(\Omega)}\leq \inf_{v_h\in S_h}\|\nabla(u-v_h)\|_{L^2(\Omega)}}$$

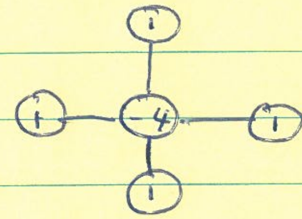
Choose $v_h=I_h u$ (interpolation)
$$\|\nabla(u-I_hu)\|_{L^2(\Omega)}\leq Ch$$
$$\|u-I_hu\|_{L^2(\Omega)}\leq Ch^2$$

$\Rightarrow$
$$\|\nabla e_h\|_{L^2(\Omega)}=O(h)$$
$$\|e_h\|_{L^2(\Omega)}=O(h^2) \quad \text{(Aubin-Nitsche trick).}$$

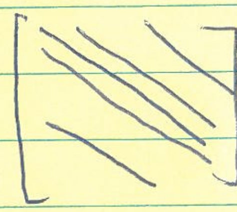
Examples of triangular meshes

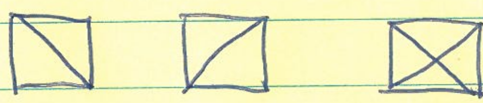
$\Omega = (0,1) \times (0,1)$





A :





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Lecture 15

FiEM for parabolic problems
— MOL approach

- Weak formulation
- FiEM approximation
- Stability
- Comments for error analysis

Weak/Galerkin formulation

$$\begin{cases} u_t = \Delta u + f & \text{in } \Omega \times I \\ u = 0 & \text{on } \partial\Omega \times I \\ u(\cdot, 0) = u^0 \end{cases}$$

Ω polygonal domain in $\mathbb{R}^2$. $I = [0, T]$

Stability estimates:

$$\|u(\cdot, t)\|_{L^2(\Omega)} \le C \|u^0\|_{L^2(\Omega)}$$

$$\|u_t(\cdot, t)\|_{L^2(\Omega)} \le \frac{C}{t} \|u^0\|_{L^2(\Omega)}$$

Weak formulation:

Find $u(t) \in H_0^1(\Omega)$, $t \in I$ such that

$$\begin{cases} (u_t, v) + a(u(t), v) = (f(t), v) & \forall v \in H_0^1(\Omega) \\ & t \in I \\ u(0) = u^0 \end{cases}$$

FEM approximation (MOL approach)

Given $\mathcal{T}_h$ for Ω, $S_h' = \text{span}\{\varphi_1, \cdots, \varphi_J\}$
(linear FE space)

- Find $u_h(t) \in S_h'$, $t \in I$, such that

$$\begin{cases} (\dot{u}_h(t), v_h) + a(u_h(t), v) = (f(t), v_h) \quad \forall v_h \in S_h', \ t \in I \\ (u_h(0), v_h) = (u^0, v_h) \quad \forall v_h \in S_h' \end{cases}$$

$\nwarrow$ L^2 projection

- matrix form

$$u_h(t) = \sum_{j=1}^{J} u_j(t) \varphi_j(x, y)$$

$$\Rightarrow \sum_{j=1}^{J} \dot{u}_j(t) (\varphi_j, \varphi_k) + \sum_{j=1}^{J} u_j a(\varphi_j, \varphi_k) = (f(t), \varphi_k) \quad k = 1, \cdots, J$$

$$\Rightarrow B\dot{\vec{u}} + A\vec{u} = \vec{b}$$

$b_{kj} = (\varphi_j, \varphi_k)$ (mass matrix)

$a_{kj} = a(\varphi_j, \varphi_k)$ (Stiffness matrix)

$b_k = (f, \varphi_k)$

B, A are symmetric and positive definite.

$\kappa(B) = O(1)$, $\kappa(A) = O(h^{-2})$ for uniform $\mathcal{T}_h$.

Stability estimate: ($f \equiv 0$)

Taking $v_h = u_h(t)$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u_h(t)\|^2_{L^2(\Omega)} + a(u_h(t), u_h(t)) = 0$$

$$\Rightarrow \|u_h(t)\|^2_{L^2(\Omega)} + 2\int_0^t a(u_h(s), u_h(s))\,ds = \|u_h(0)\|^2_{L^2(\Omega)}$$

$$\Rightarrow \|u_h(t)\|_{L^2(\Omega)} \le \|u_h(0)\|_{L^2(\Omega)} \le \|u^0\|_{L^2(\Omega)} \quad \#$$

- error analysis

Recall
$$(\dot{u}, v) + a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$
$$(\dot{u}_h, v_h) + a(u_h, v_h) = (f, v_h) \quad \forall v_h \in S_h^1$$

$$\Rightarrow (\dot{e}_h, v_h) + a(e_h, v_h) = 0 \quad \forall v_h \in S_h$$

(orthogonality)

which is different from $a(e_h, v_h) = 0$ for elliptic problem.

Key steps in the error analysis:

- introduce an intermediate approximation $\tilde{u}_h(t) \in S_h$, $t \in I$:
$$a(u(t) - \tilde{u}_h(t), v_h) = 0 \quad \forall v_h \in S_h^1,\ t \in I$$
i.e., $\tilde{u}_h(t)$ is the projection of $u(t)$ on S_h^1 in the energy norm.

- define a dual problem

For any $t \in I$: $\varphi_h(s) \in S_h^1$, $s \in (0,t)$:

$$\begin{cases} -(\dot{\varphi}_h(s), v_h) + a(\varphi_h(s), v_h) = 0 \\ \qquad\qquad\qquad \forall\, v_h \in S_h^1,\ s \in (0,t) \\ \varphi_h(t) = \tilde{e}_h(t) \equiv \tilde{u}_h(t) - u_h(t) \end{cases}$$

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