Exercise 2.1

(i) The Fourier sine series expansion

$$u^0(x) = \sum a_m \sin m\pi x$$

has coefficients

$$\begin{aligned}
a_m &= 2\int_0^1 u^0(x)\sin m\pi x \mathrm{d}x \\
&= 2\int_0^{\frac{1}{2}} 2x \sin m\pi x \mathrm{d}x + 2\int_{\frac{1}{2}}^1 (2-2x)\sin m\pi x \mathrm{d}x \\
&= 2\left\{2x\left(\frac{-\cos m\pi x}{m\pi}\right) - 2\left(\frac{-\sin m\pi x}{m^2\pi^2}\right)\right\}_0^{\frac{1}{2}} \\
&+ 2\left\{(2-2x)\left(\frac{-\cos m\pi x}{m\pi}\right) - (-2)\left(\frac{-\sin m\pi x}{m^2\pi^2}\right)\right\}_{\frac{1}{2}}^1 \\
&= \frac{8}{m^2\pi^2}\sin\frac{m\pi}{2}.
\end{aligned}$$

(ii) $$\int_{2p}^{2p+2}\frac{1}{x^2}\mathrm{d}x = \left[-\frac{1}{x}\right]_{2p}^{2p+2} = \frac{1}{2p} - \frac{1}{2p+2} = \frac{2}{2p(2p+2)} = \frac{2}{(2p+1)^2-1} > \frac{2}{(2p+1)^2}.$$

Hence $$\sum_{p_0}^{\infty}\frac{1}{(2p+1)^2} < \tfrac{1}{2}\sum_{p_0}^{\infty}\int_{2p}^{2p+2}\frac{\mathrm{d}x}{x^2} = \tfrac{1}{2}\int_{2p_0}^{\infty}\frac{\mathrm{d}x}{x^2} = \frac{1}{4p_0}.$$

(iii) The remainder after the term in m has magnitude less than

$$\sum_{m+1}^{op}\frac{8}{m^2\pi^2}|\sin\frac{m\pi}{2}||\sin m\pi x| < \sum_{p_0}^{\infty}\frac{8}{(2p+1)^2\pi^2} < \frac{2}{\pi^2 p_0}, \text{ if } m = 2p_0 - 1.$$

The remainder $< \frac{1}{1000}$ if $p_0 > \frac{2000}{\pi^2} = 202.6$. This is satisfied if $m = 405$.