Exercise 2.2

Using Taylor series with remainder,

$$\begin{aligned} e^{-K^2\Delta t} &= 1-(K^2\Delta t)+\tfrac{1}{2}(K^2\Delta t)^2 e^{-\xi}, \quad 0<\xi<K^2\Delta t \\ &= 1-K^2\Delta t+\tfrac{1}{2}K^4(\Delta t)^2\alpha \quad \text{ where } 0<\alpha<1 \end{aligned}$$

and

$$\begin{aligned} \sin^2(\tfrac{1}{2}K\Delta x) &= \tfrac{1}{2}(1-\cos K\Delta x) \\ &= 0+\tfrac{1}{4}(K\Delta x)^2+0+\left(-\tfrac{1}{2}\right)\tfrac{1}{24}(K\Delta x)^4\cos\eta, \quad 0<\eta<K\Delta x \\ &= \tfrac{1}{4}K^2(\Delta x)^2-\tfrac{\beta}{48}K^4(\Delta x)^4 \quad \text{ where } 0<\beta<1 \end{aligned}$$

Hence

$$\begin{aligned} &\left|1-4\nu\sin^2(\tfrac{1}{2}K\Delta x)-e^{-K^2\Delta t}\right| \\ &= \left|1-4\nu\left[\tfrac{1}{4}K^2(\Delta x)^2-\tfrac{\beta}{48}K^4(\Delta x)^4\right]-\left[1-K^2\Delta t+\tfrac{\alpha}{2}K^4(\Delta t)^2\right]\right| \\ &= \left|K^4(\Delta t)^2\left[\tfrac{\beta}{12\nu}-\tfrac{\alpha}{2}\right]\right| \\ &\le C(\nu)K^4(\Delta t)^2 \quad \text{ where } C(\nu)=\left|\tfrac{\beta}{12\nu}-\tfrac{\alpha}{2}\right|. \end{aligned}$$

When $\nu=\frac{1}{4}$ this gives $C(\nu)=C(\frac{1}{4})=\frac{1}{2}$

¿From Exercise 1, $\sum|a_m|\le\frac{8}{\pi^2}\cdot\frac{1}{4p_0}\le\frac{0.01}{4}$ when $p_0\ge 82$

and $\displaystyle\sum_1^{2p_0-1}|a_m m^4|\le\frac{8}{\pi^2}[1^2+3^2\ldots+(2p-1)^2]=\frac{8}{3\pi^2}p_0(2p_0+1)(2p_0-1).$

Hence from (2.62)
$$\begin{aligned} |e_j^n| &\le \frac{1}{200}+1\cdot\tfrac{1}{2}\pi^4\frac{8}{3\pi^2}\cdot 82\cdot 165\cdot 163\Delta t \\ &\le \frac{1}{100} \end{aligned}$$

if $\displaystyle\Delta t\le\frac{1}{200}\frac{6\pi^2}{\pi^4\cdot 8\cdot 82\cdot 165\cdot 163}=1.7\times 10^{-10}$