Exercise 2.3

Expanding left hand side gives

$$u_t + \tfrac{1}{2}\Delta t\ u_{tt} + \cdots$$

and the right hand side

$$\frac{2}{\Delta x_{j-1} + \Delta x_j}\left\{ [u_x + \tfrac{1}{2}\Delta x_j u_{xx} + \tfrac{1}{6}(\Delta x_j)^2 u_{xxx} + \tfrac{1}{24}(\Delta x_j)^3 u_{xxxx} + \ldots] \right.$$
$$\left. -[u_x - \tfrac{1}{2}\Delta x_{j-1} + \tfrac{1}{6}(\Delta x_{j-1})^2 u_{xxx} - \tfrac{1}{24}(\Delta x_{j-1})^3 u_{xxxx} + \ldots]\right\}$$
$$= u_{xx} + \tfrac{1}{3}(\Delta x_j\ -\ \Delta x_{j-1})u_{xxx} + \tfrac{2}{24}((\Delta x_j)^2 + (\Delta x_{j-1})^2 - \Delta x_j \Delta x_{j-1})u_{xxxx}\ldots.$$

So

$$T_j^n\ =\ \tfrac{1}{2}\Delta t u_{tt} - \tfrac{1}{3}(\Delta x_j - \Delta x_{j-1})u_{xxx} - \tfrac{1}{12}[\Delta x_j^2 + \Delta x_{j-1}^2 - \Delta x_j \Delta x_{j-1}]u_{xxxx}\ldots,$$

and

$$|T_j^n| \le T\ =\ \tfrac{1}{2}\Delta t M_{tt} + \tfrac{1}{3}\alpha(\Delta x)^2 M_{xxx} + \tfrac{1}{12}(\Delta x)^2[1 + \alpha(\Delta x)^2]M_{xxxx}.$$

Then writing $\mathrm{e}_j^n = U_j^n - u(x_j t_n)$ we get

$$\mathrm{e}_j^{n+1}\ =\ \mathrm{e}_j^n\left[1 - \frac{2\Delta t}{\Delta x_{j-1} + \Delta x_j \Delta x_{j-1}}\left(\frac{1}{\Delta x_{j-1}} + \frac{1}{\Delta x_j}\right)\right]$$
$$+\frac{2}{(\Delta x_{j-1} + \Delta x_j)\Delta x_j}\mathrm{e}_j^n + \frac{2}{(\Delta x_{j-1} + \Delta x_j)\Delta x_{j-1}}\mathrm{e}_{j-1}^n + \Delta t\ T_j^n.$$

The sum of the coefficients on the right is 1, and they are all non-negative provided

$$\Delta t \le \tfrac{1}{2}\Delta x_{j-1}\Delta x_j.$$

Then

$$\left|\mathrm{e}_j^n\right| \le n\Delta t\ T \le t_F T \text{ in the usual way.}$$

Exercise 2.4

Suppose $0 \le \mathrm{e}_{j-1} < 1$.

Then

$$b_j - a_j < b_j - a_j \mathrm{e}_{j-1} \le b_j \qquad \text{since} \;\; a_j > 0$$

and since $b_j - a_j > c_j$ this gives

$$c_j < b_j - a_j \mathrm{e}_{j-1} \le b_j$$

giving

$$0 < \mathrm{e}_j = \frac{c_j}{b_j - a_j \mathrm{e}_{j-1}} < 1$$

Since $\mathrm{e}_0 = 0$ this shows by induction that $0 < \mathrm{e}_{j-1} < 1$ for $j = 1, 2, \ldots, J-1$.

In the same way, if $|\mathrm{e}_{j-1}| \le 1$ then

$$\begin{aligned} |b_j - a_j \mathrm{e}_{j-1}| &\ge \big||b_j| - |a_j|\,|\mathrm{e}_{j-1}|\big| \\ &\ge |b_j| - |a_j| \\ &\ge |c_j| \end{aligned}$$

and hence $|\mathrm{e}_j| \le 1$.

So by induction if $|\mathrm{e}_0| \le 1$ then $|\mathrm{e}_j| \le 1$ for $j = 1, 2, \ldots J-1$.