Exercise 2.7

Expansion of left hand side:–

$$U_t + \tfrac{1}{2}\Delta t\ U_{tt} + \tfrac{1}{6}(\Delta t)^2 U_{ttt} + \ldots .$$

For the right hand side

$$U_{j+1}^n = [u + \tfrac{1}{2}\Delta x\ u_x + \tfrac{1}{2}(\tfrac{1}{2}\Delta x)^2 u_{xx} + \tfrac{1}{6}(\tfrac{1}{2}\Delta x)^3 u_{xxx} + \tfrac{1}{24}(\tfrac{1}{2}\Delta x)^4 u_{xxxx}\ldots]_{j+\frac{1}{2}}^n.$$

U_j^n is the same, with alternating signs, so

$$\frac{U_{j+1}^n - U_j^n}{\Delta x} = [u_x + \tfrac{1}{24}(\Delta x)^2 u_{xxx} + \ldots .]_{j+\frac{1}{2}}$$

Now expanding about (x_j, t_n),

$$\begin{aligned}\left(\frac{U_{j+1}^n - U_j^n}{\Delta x}\right) p_{j+\frac{1}{2}} = {} & [u_x + \tfrac{1}{24}(\Delta x)^2 u_{xxx}\ldots]p \\ & + \tfrac{1}{2}\Delta x\ \frac{\partial}{\partial x}\left\{\left[u_x + \tfrac{1}{24}(\Delta x)^2 u_{xxx}\ldots\right] p\right\} \\ & + \ldots .\end{aligned}$$

The required result is the difference of two terms of this form, so we need only the terms of odd degree, giving

$$\begin{aligned}\frac{2}{\Delta x}\Big\{\tfrac{1}{2}\Delta x\ \frac{\partial}{\partial x}\{[u_x + \tfrac{1}{24}(\Delta x)^2 u_{xxx}\ldots]p\} + \tfrac{1}{6}(\tfrac{1}{2}\Delta x)^3 \frac{\partial^3}{\partial x^3}\{[u_x + \tfrac{1}{24}(\delta x)^2 u_{xxx}\ldots]p\} + \ldots \\ = u_{xx} + \tfrac{1}{24}(\delta x)^2 \frac{\partial}{\partial x}(p u_{xxx})\Big\} + \tfrac{1}{24}(\delta x)^2 \frac{\partial^3}{\partial x^3}(p u_x) + \ldots .\end{aligned}$$

Hence the truncation error, the difference between the left and right hand sides, becomes

$$T_j^n = \tfrac{1}{2}\Delta t\ u_{tt} - \tfrac{1}{24}(\delta x)^2 \left\{\frac{\partial}{\partial x}(p\frac{\partial^3 u}{\partial x^3}) + \frac{\partial^3}{\partial x^3}(p\frac{\partial u}{\partial x})\right\}.$$

So $\left|T_j^n\right| \le T = \tfrac{1}{2}\Delta t\ M_{2t} + \tfrac{1}{24}(\Delta x)^2 M_x^*$

where
$$\begin{aligned} M_{2t} &= \max|P u_{tt}| \\ M_x^* &= \max|(p u_{xxx})_x + (p u_x)_{xxx}|\end{aligned}$$

Writing $\mathrm{e}_j^n = U_j^n - u(x_j t_n)$ gives in the usual way

$$\begin{aligned} \mathrm{e}_j^{n+1} &= \mathrm{e}_j^n \left[1 - \frac{\Delta t}{(\Delta x)^2}(p_{j+\frac{1}{2}} + p_{j-\frac{1}{2}}) \right] + \mathrm{e}_{j+1}^n \frac{\Delta t}{(\Delta x)^2} p_{j+\frac{1}{2}} \\ &\qquad + \mathrm{e}_{j-1}^n \frac{\Delta t}{(\Delta x)^2} p_{j-\frac{1}{2}} - \Delta t \; T_j^n \end{aligned}$$

and then, if $\epsilon^n = \max_j \left| \mathrm{e}_j^n \right|$,

$$\epsilon^{n+1} \leq \epsilon^n + \Delta t \; T, \text{ provided } p(x) > 0, \text{ and } 2p(x)\frac{\Delta t}{(\Delta x)^2} \leq 1$$

so that all the coefficients of e on the right are non-negative. Hence

$$\begin{aligned} \left| U_j^n - u(x_j, t_n) \right| \leq \epsilon^n &\leq n \Delta t \; T \\ &\leq t_F T \quad \text{ on } 0 \leq t \leq t_F. \end{aligned}$$