HW #1 Prob 2.1

Due Thursday

Lecture 1

Math 783

Text: Numerical Solution of Partial Differential Equations
by K.W. Morton and D.F. Mayers
Cambridge University Press, 1994

Scientific Computing

physical motivation
mathematical background
numerical analysis techniques
Computer knowledge

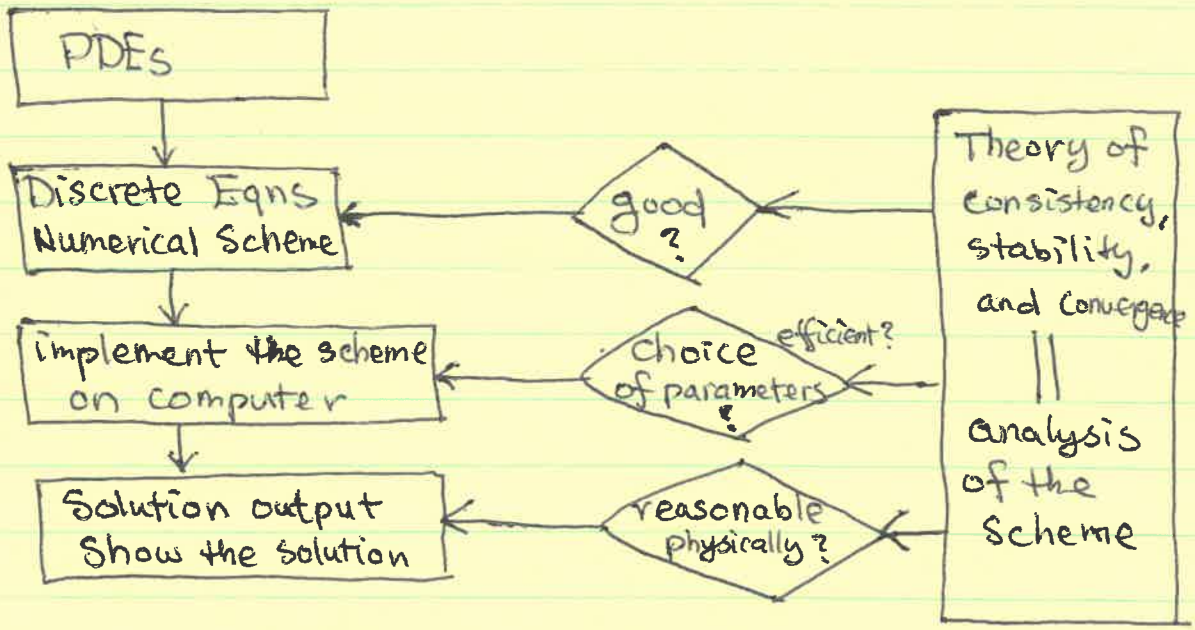
$\downarrow$

find the numerical solution of the practical problem

$\downarrow$

Show and illustrate your numerical solution

Numerical Analysics or Computional Mathematics



Consistency : Does the numerical scheme or the discrete equation approach close enough to the original, continuous PDE ?

Stability : Does the numerical solution stay bounded during the course of the solution of the discrete equation ?

Convergence : Does the numerical solution of the discrete equation approach close enough to the exact solution (often is unknown) of the underlying continuous PDE ?

About this course

	Math 783	Math 796 (Fall, 96)
PDEs	linear parabolic, hyperbolic, elliptic (mainly)	nonlinear, higher dimensions
schemes	basic	advanced
discretization	FD + FE	spectral
Solution method	basic, simple iteration	multigrid, preconditioning technique, conjugate gradient

Computer Knowledge:

C and/or Fortran 77 desired

Rounding error:

Four types of data:

integer 1, 0, −1, ...

real = single, double 1.000000 (7 digit) 1.00000000000000 (14 ~ 16 digits)

logical = 'true' (1), 'false' (0)

character 'hi', 'you'

Let us assume single precision (7 digits)

real number 1/3 0.333333 (computer expression)

difference $= \frac{1}{3} - 0.333333 = 0.000000333\cdots$

1.0 $\underbrace{1.000001}_{\text{tail}}$ or 0.999999

x, $\tilde{x}$ (computer expression)

Let $x = \tilde{x} + \varepsilon_x$, $y = \tilde{y} + \varepsilon_y$, $\varepsilon_x \approx 10^{-7}$, $\varepsilon_y \sim 10^{-7}$

$x \pm y$: $\tilde{x} \pm \tilde{y} = x \pm y - (\varepsilon_x \pm \varepsilon_y) \approx x + y \pm 2\times 10^{-7}$

$x \cdot y$: $\tilde{x} \cdot \tilde{y} = x \cdot y + x \cdot \varepsilon_y + y \cdot \varepsilon_x \approx xy + (|x| + |y|) \times 10^{-7}$

$$\frac{x}{y}: \quad \frac{\tilde{x}}{\tilde{y}} = \frac{x-\varepsilon_x}{y-\varepsilon_y} = \frac{1}{y}\,\frac{x-\varepsilon_x}{1-\varepsilon_y/y}$$

$$\approx \frac{1}{y}\cdot(x-\varepsilon_x)(1+\varepsilon_y/y)$$

$$= \frac{x}{y} - \frac{\varepsilon_x}{y} + \frac{x}{y}\cdot\frac{\varepsilon_y}{y} + O(\varepsilon_x\varepsilon_y)$$

$$= \frac{x}{y} + \frac{10^{-7}}{|y|} + \left|\frac{x}{y}\right|\cdot\frac{10^{-7}}{|y|} \qquad ? \text{ when } |y|\approx 0 ?$$

Conclusion: avoid the situation of y ~~being~~ being small when compute $\frac{x}{y}$.

Chapter 2 Parabolic Equations in One Space Variable

§§2.1–2.4 An Explicit Scheme for the Model Problem

Linear parabolic equation

PDE: $e(x,t)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u + d(x,t)$

where e and $a > 0$. IBVP:

IC. $u(x,0) = u^0(x)$

BC: at $x=0$. $\alpha_0(t)u + \alpha_1(t)\frac{\partial u}{\partial x} = \alpha_2(t)$

$(\alpha_0 \geq 0,\ \alpha_1 \leq 0,\ \alpha_0 - \alpha_1 > 0)$

at $x=1$ $\beta_0(t)u + \beta_1(t)\frac{\partial u}{\partial x} = \beta_2(t)$

$(\beta_0 \geq 0,\ \beta_1 \geq 0,\ \beta_0 + \beta_1 > 0)$

<u>Analytical Method</u>: Seperation of variables

$\oplus$ Fourier series expansion

<u>Model Problem</u>: IBVP

$$\begin{cases} u_t = u_{xx} & t_F > t > 0,\quad 0 < x < 1 \\ u(0,t) = u(1,t) = 0 & t_F > t > 0 \\ u(x,0) = u^0(x) & 0 < x < 1 \end{cases}$$

$$\begin{cases} u(x,t) = \sum_{m=1}^{\infty} a_m e^{-(m\pi)^2 t} \sin(m\pi x) \\ a_m = 2\int_0^1 u^0(x)\sin(m\pi x)\,dx \end{cases}$$

Mesh (Grid):

Spatial domain $x \in [0,1]$, $\Delta x = \frac{1}{J}$, $x_j = j\Delta x$

time domain $t \in [0, t_F]$ $\Delta t = \frac{1}{N}$, $t_n = n\Delta t$

mesh points (x_j, t_n) $j = 0, 1, \cdots, J$, $n = 0, 1, \cdots, N$

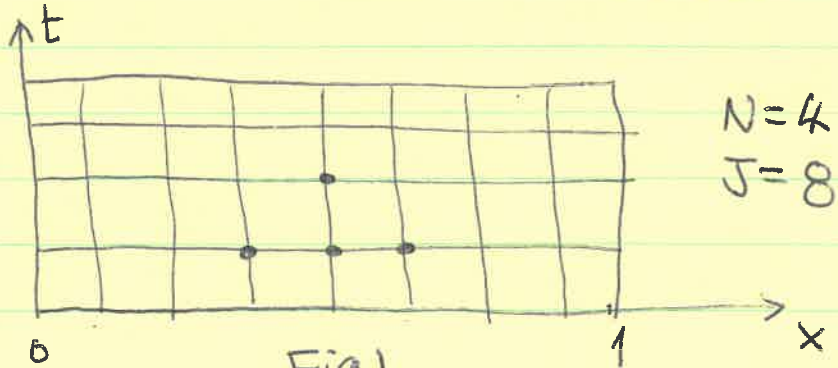


Fig 1

Objective a. Seek approximations of the solution at the mesh points

$$U_j^n \approx u(x_j, t_n)$$

b. $n=0$, $U_j^0 = u^0(x_j)$, $j = 0, \cdots, J$ (known)

c. $U_0^n = 0$, $U_J^n = 0$, $n = 1, \cdots, N$ (known)

d. $U_j^n = ?$ $j = 1, 2, \cdots, J-1$, $n = 1, 2, \cdots, N$

FD Discretization

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0<x<1, \ 0<t<t_F$$

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \quad \text{for } 0<x<1, \ 0<t<t_F$$

$$\frac{\partial u}{\partial t}(x_j, t_n) = \frac{\partial^2 u}{\partial x^2}(x_j, t_n), \quad j = 1, 2, \cdots, J-1, \ n = 1, 2, \cdots, N-1.$$

From Calculus,

$$\lim_{\varepsilon \to 0} \frac{u(x_j, t_n+\varepsilon) - u(x_j, t_n)}{\varepsilon} = \frac{\partial u}{\partial t}(x_j, t_n)$$

$$\frac{u(x_j, t_n+\Delta t) - u(x_j, t_n)}{\Delta t} \approx \frac{\partial u}{\partial t}(x_j, t_n) \text{ if } \Delta t \text{ small}$$

$$\frac{u(x_{j+1},t_n)-2u(x_j,t_n)+u(x_{j-1},t_n)}{(\Delta x)^2} \approx \frac{\partial^2 u}{\partial x^2}(x_j,t_n) \text{ if } \Delta x \ll 1$$

$$\frac{\partial u}{\partial t}(x_j,t_n) = \frac{\partial^2 u}{\partial x^2}(x_j,t_n) \Longrightarrow$$

$$\frac{u(x_j,t_{n+1})-u(x_j,t_n)}{\Delta t} \approx \frac{u(x_{j+1},t_n)-2u(x_j,t_n)+u(x_{j-1},t_n)}{(\Delta x)^2}$$

Replacing $u(x_j,t_n)$ by U_j^n:

$$\frac{U_j^{n+1}-U_j^n}{\Delta t} = \frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{(\Delta x)^2}$$

or

$$\boxed{U_j^{n+1} = U_j^n + \nu\left[U_{j+1}^n - 2U_j^n + U_{j-1}^n\right] \qquad \nu = \frac{\Delta t}{\Delta x^2}}$$ (Ex Sch)

Procedure: $U_j^n = ?$ $\quad j=1,2,\cdots,J-1,\ n=1,2,\cdots N$

① $n=0$, U_j^0, $j=0,1,\cdots,J$ known

See Fig 1

$$\left.\begin{array}{l} \text{Ex Sch (r.h.s. known)} \Longrightarrow U_j^1,\ j=1,2,\cdots,J-1 \\ \text{BCs} \Longrightarrow U_0^1,\ U_J^1 \end{array}\right\} n=1 \text{ known}$$

② $n=1$, U_j^1 $j=0,1,\cdots,J$ known

$$\left.\begin{array}{l} \text{Ex Sch (r.h.s. known)} \Longrightarrow U_j^2,\ j=1,2,\cdots,J-1 \\ \text{BCs} \Longrightarrow U_0^2,\ U_J^1 \end{array}\right\} n=2 \text{ known}$$

⋮

Explicit