Hour 13

Lecture 12

Polar coordinates §2.16

Consider the parabolic equation

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \subset \mathbb{R}^3 \\ u|_{\partial\Omega} \text{ given} \end{cases}$$

in cylindrical or spherical coordinates.

$$\frac{\partial u}{\partial t} = \frac{1}{r^\alpha}\frac{\partial}{\partial r}\left(r^\alpha \frac{\partial u}{\partial r}\right)$$

$$\text{or} \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{\alpha}{r}\frac{\partial u}{\partial r}.$$

$\alpha = 1$: cylindrical
$\alpha = 2$: spherical

Fourier series $\Rightarrow$

$$\frac{\partial u}{\partial t} = \frac{1}{r^\alpha}\frac{\partial}{\partial r}\left(r^\alpha \frac{\partial u}{\partial r}\right) - \frac{n^2}{r^2}u$$

<u>IVP</u>:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r}\right) & 0 \le r \le 1 \\ u(1,t) = g(t), \quad u(0,t) \text{ bounded} \\ u(r,0) = u^0(r) \end{cases}$$

<u>FD Discretization</u>:

$$r_0 = 0 < r_1 < \cdots < r_J = 1$$

$$r_j = j\Delta r, \quad \Delta r = \frac{1}{J}$$

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\Delta t} = \dfrac{2}{r_j^2 (r_{j+1} - r_{j-1})} \left[r_{j+1/2}^2 \dfrac{U_{j+1}^n - U_j^n}{\Delta r} - r_{j-1/2}^2 \dfrac{U_j^n - U_{j-1}^n}{\Delta r} \right] \\ \qquad\qquad j = 1, 2, \cdots, J-1 \\ U_J^{n+1} = g(t_{n+1}) \end{cases}$$

J eqns for $J+1$ unknowns ?

Pole Conditions:

Assume that $u(r,t)$ is sufficiently smooth.

$$u(r,t) = u(0,t) + \frac{\partial u}{\partial r}(0,t)\, r + \frac{\partial^2 u}{\partial r^2}(0,t) \frac{r^2}{2!} + \cdots$$

$$= \sum_{j=0}^{\infty} \frac{r^j}{j!} \frac{\partial^j u}{\partial r^j}(0,t) \qquad \text{for } r \text{ small}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} \right)(r,t) = \sum_{j=0}^{\infty} \frac{r^{j-2}}{j!} (j+1) j \, \frac{\partial^j u}{\partial r^j}(0,t)$$

$$\frac{\partial u}{\partial t} = \sum_{j=0}^{\infty} \frac{r^j}{j!} \frac{\partial}{\partial t}\left(\frac{\partial^j u}{\partial r^j}(0,t) \right) \qquad \|$$

$$= \sum_{j=-2}^{\infty} \frac{(j+2)(j+3)\, r^j}{(j+2)!} \frac{\partial^{j+2} u}{\partial r^{j+2}}(0,t)$$

$$\sum_{j=0}^{\infty} \frac{r^j}{j!} \frac{\partial^{j+1} u}{\partial t \partial r^j}(0,t) = \sum_{j=-2}^{\infty} \frac{(j+2)(j+3)\, r^j}{(j+2)!} \frac{\partial^{j+2} u}{\partial r^{j+2}}(0,t)$$

for r small

$j=-2$: $\quad 0 = 0 \qquad O(r^{-2})$

$j=-1$: $\quad 0 = 2\frac{\partial u}{\partial r}(0,t) \qquad O(r^{-1})$

$j \geq 0$: $\quad \frac{1}{j!}\frac{\partial^{j+1} u}{\partial t \partial r^j}(0,t) = \frac{(j+2)(j+3)}{(j+2)!}\frac{\partial^{j+2} u}{\partial r^{j+2}}(0,t) \qquad O(r^j)$

$$\boxed{\frac{\partial u}{\partial r}(0,t) = 0}$$

Tech I Direct Expansion (?)

$$\frac{\partial u}{\partial t}(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$$

$$\lim_{r\to 0} \frac{\partial u}{\partial r}(r,t_n) = \lim_{r \to 0} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)(r,t_n)$$

$$\lim_{r\to 0} \frac{\partial u}{\partial t}(r,t_n) = \frac{\partial u}{\partial t}(0,t_n) \approx \frac{U_0^{n+1} - U_0^n}{\Delta t}$$

$$\lim_{r\to 0} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)(r,t_n) = ?$$

$\frac{\partial u}{\partial r}(0,t)=0$!

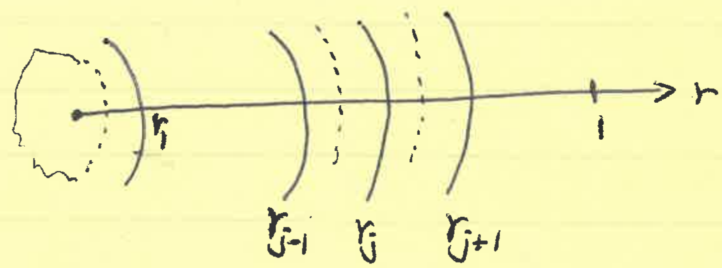
$$\begin{cases} \lim_{r\to 0} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)(r,t_n) = \boxed{3}\, \frac{\partial^2 u}{\partial r^2}(0,t_n) \\ u(\Delta r, t_n) = u(0,t_n) + \frac{\partial^2 u}{\partial r^2}(0,t_n)\frac{\Delta r^2}{2!} + O(\Delta r^3) \end{cases}$$

$$\lim_{r\to 0} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)(r,t_n) = \frac{u(\Delta r,t_n) - u(0,t_n)}{\Delta r^2}\cdot 6 + O(\Delta r)$$

$$\boxed{\frac{U_0^{n+1} - U_0^n}{\Delta t} = \frac{U_1^n - U_0^n}{\Delta r^2}} \qquad O(\Delta r)$$

Tech 2 (Physical motivation)

u: temperature



$r^2 \frac{\partial u}{\partial r} \propto$ Total heat flux across surface at r

= heat flux ⊗ area of surface at r

$r^2 \frac{\partial u}{\partial r}\Big|_{j+\frac12} - r^2 \frac{\partial u}{\partial r}\Big|_{j-\frac12} \propto$ increase of temperature $\frac{\partial u}{\partial t}$

⊗ volume of the domain

between $r = r_{j+\frac12}$ and $r = r_{j-\frac12}$

$$r^2 \frac{\partial u}{\partial r}\Big|_{j+\frac12} - r^2 \frac{\partial u}{\partial r}\Big|_{j-\frac12} = \frac{\partial u}{\partial t}(r_j, t)\cdot \frac{1}{3}\left(r_{j+\frac12}^3 - r_{j-\frac12}^3\right)$$

$$\begin{cases} \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{3}{r_{j+\frac12}^3 - r_{j-\frac12}^3}\left[r_{j+\frac12}^2 \frac{U_{j+1}^n - U_j^n}{\Delta r} - r_{j-\frac12}^2 \frac{U_j^n - U_{j-1}^n}{\Delta r}\right] \\ U_J^{n+1} = g(t_{n+1}) \end{cases}$$

$j = 1, 2, \cdots, J-1$

$$r^2 \frac{\partial u}{\partial r}\Big|_{1/2} = \frac{\partial u}{\partial t}\Big|_{r=0} \cdot \frac{1}{3} r_{1/2}^3$$

$$\boxed{\frac{U_0^{n+1} - U_0^n}{\Delta t} = \frac{6}{\Delta r} \cdot \frac{U_1^n - U_0^n}{\Delta r}}$$

Tech 3 (Reformulation, Huang & Sloan, J. Comput. Phys. 1994)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r}\right) & 0 < r < 1 \\ u(1,t) = g(t), \quad \dfrac{\partial u}{\partial r}(0,t) = 0 \\ u(r,0) = u^0(r) \end{cases}$$

① $\dfrac{U_1^{n+1} - U_0^{n+1}}{\Delta r} = 0$

② staggered mesh in space.

Advantages

① Standard discretization methods, such as FD, FE, and spectral methods, can be applied without any increase in complexity.

② works well for nonlinear problems.

Q ① which method is the best for time dependent problems?

2. How serious is the CFL condition if explicit schemes is used ?
3. If spectral method is used, is it possible to find a good FD preconditioner ?
4. Any better method ?
5. Higher Dimensional problems ?