Hoar 14, 15

Mesh adaptation / movement and the equidistribution principle for numerical solution of 1D parabolic PDEs

Burgers' Equation:

$$\begin{cases} u_t = \varepsilon u_{xx} - u u_x & 0 < x < 1,\ t > 0 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \sin(2\pi x) + \frac{1}{2}\sin(\pi x) \end{cases}$$

ε : physical parameter

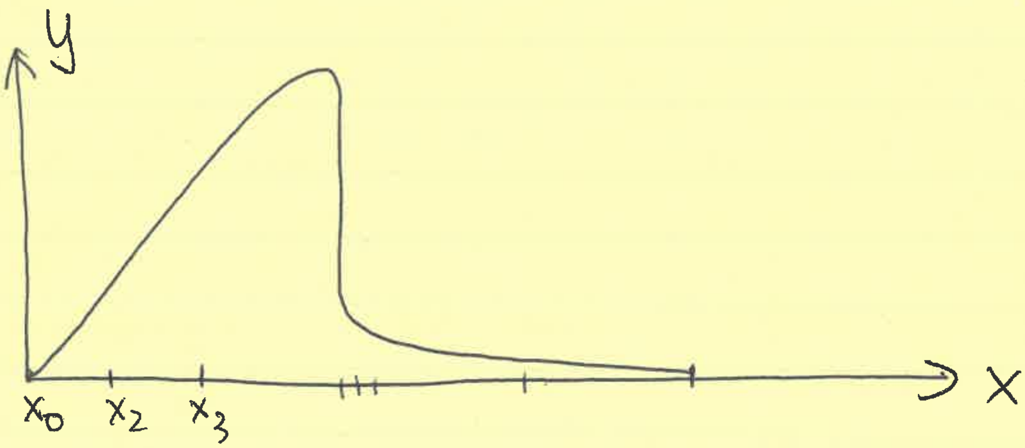
$x_0 < x_1 < \cdots < x_J$

1. Why to need a non-uniform mesh?

   $\varepsilon = 10^{-4}$. Show the exact soln and computed soln with uniform mesh.

2. How to generate a non-uniform mesh adapting the physical soln?

   The equidistribution principle.



Measure: $|u_x(x_{j+1/2}, t)| \cdot (x_{j+1} - x_j)$

$$|u_x(x_{j+1/2}, t)| \, (x_{j+1} - x_j) = \text{Constant.}$$

$$j = 0, 1, 2, \cdots, J-1$$

Singularity

$$\sqrt{u_x^2(x_{j+1/2}, t) + 1} \, (x_{j+1} - x_j) = \text{Const.}$$

Solution

$$\begin{cases} \sqrt{u_x^2(x_{1/2}, t) + 1} \, (x_1 - x_0) - \sqrt{u_x^2(x_{3/2}, t) + 1} \, (x_2 - x_1) = 0 \\ \quad - - - - \\ \sqrt{u_x^2(x_{J-3/2}, t) + 1} \, (x_{J-1} - x_{J-2}) - \sqrt{\quad} \, (x_J - x_{J-1}) = 0 \\ \qquad (J-1) \text{ eqns} \\ x_0 = 0, \quad x_J = 1 \end{cases}$$

Continuous Representation and Coordinate transformation

$$\begin{cases} x = x(\xi, t), \quad x(0,t) = 0, \quad x(1,t) = 1 \\ t = \tau \end{cases}$$

$$x_j = x(\xi_j, t), \quad \xi_j = \frac{j}{J}$$

$$\sqrt{\left(\frac{\partial u}{\partial x}(x(\xi_{j+\frac{1}{2}}), t)\right)^2 + 1}\ \frac{\left(x(\xi_{j+1}, t) - x(\xi_j, t)\right)}{\xi_{j+1} - \xi_j} = \frac{\text{Const.}}{\frac{1}{J}}$$

$$M(\xi, t) \equiv \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + 1}\ \frac{\partial x}{\partial \xi} = \text{Const.} \cdot J$$

$$M = M(x, u)$$

$$\begin{cases} \begin{cases} \dfrac{\partial}{\partial \xi}\left[M \dfrac{\partial x}{\partial \xi}\right] = 0 \\ x(0,t) = 0, \quad x(1,t) = 1 \end{cases} \\ \dfrac{\partial u}{\partial t} - \dfrac{\frac{\partial u}{\partial \xi}}{\frac{\partial x}{\partial \xi}} \dfrac{\partial x}{\partial t} = \dfrac{1}{x_\xi}\left(\dfrac{u_\xi}{x_\xi}\right)_\xi - \dfrac{(u^2)_\xi}{2 x_\xi}. \end{cases}$$

3. Moving Mesh PDEs

$$\dot{x} = \frac{1}{\tau} \frac{\partial}{\partial \xi} \left[M \frac{\partial x}{\partial \xi} \right]$$