CG, CR, MR, GMRES, BiCG, BiCG Stab, BiCG Stab2

Preconditioning.

This approach is very robust, reliable but may not very efficient and Labor.

(B) change the scheme Approximate Fatorization.

This approach is very efficient, if works.

find something between implicit and explicit.

3月1日

§§ 3.2 ⊕ §3.3. ADI

AF: approximate factorization

$$\begin{cases} u_t = \Delta u & \text{in } \Omega = (0,1)\times(0,1) \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

work for { regular domains and rectangular meshes

very restrictive!!

$$u_t = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{u_{j,k}^{n+1} - u_{j,k}^{n}}{\Delta t} = \frac{1}{\Delta x^2}\delta_x^2 u_{j,k}^{n+1} + \frac{1}{\Delta y^2}\delta_y^2 u_{j,k}^{n+1}$$

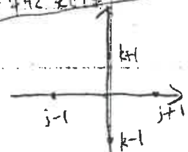
Define: increment variable

$$\Delta u_{j,k}^n := u_{j,k}^{n+1} - u_{j,k}^n = O(\Delta t)$$

$$\frac{\Delta u_{j,k}^n}{\Delta t} = \frac{1}{\Delta x^2}\delta_x^2 \Delta u_{j,k}^n + \frac{1}{\Delta y^2}\delta_y^2 \Delta u_{j,k}^n + \frac{1}{\Delta x^2}\delta_x^2 u_{j,k}^n + \frac{1}{\Delta y^2}\delta_y^2 u_{j,k}^n$$

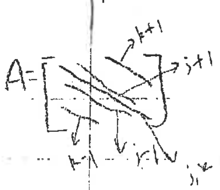
"I" means $I u_{j,k} = u_{j,k}$
可视作单位矩阵

$$\left[I - \frac{\Delta t}{\Delta x^2}\delta_x^2 - \frac{\Delta t}{\Delta y^2}\delta_y^2\right]\Delta u_{j,k}^n = \Delta t\left[\frac{1}{\Delta x^2}\delta_x^2 + \frac{1}{\Delta y^2}\delta_y^2\right]u_{j,k}^n$$



$$1 - a - b = (1-a)(1-b) - \underbrace{ab}_{\text{if small, drop}}$$

$$\approx (1-a)(1-b)$$



$$I - \frac{\Delta t}{\Delta x^2}\delta_x^2 - \frac{\Delta t}{\Delta y^2}\delta_y^2 = \left(I - \frac{\Delta t}{\Delta x^2}\delta_x^2\right)\left(I - \frac{\Delta t}{\Delta y^2}\delta_y^2\right) - \frac{\Delta t^2}{\Delta x^2 \Delta y^2}\delta_x^2\delta_y^2$$

$$\approx \left(I - \frac{\Delta t}{\Delta x^2}\delta_x^2\right)\left(I - \frac{\Delta t}{\Delta y^2}\delta_y^2\right)$$

$$\frac{\Delta t^2}{\Delta x^2 \Delta y^2}\delta_x^2\delta_y^2\Delta u_{j,k}^n = \frac{\Delta t^2}{\Delta x^2\Delta y^2}\delta_x^2\delta_y^2\left(u_{j,k}^{n+1}-u_{j,k}^n\right)$$

$$\approx \frac{\Delta t^2}{\Delta x^2\Delta y^2}\delta_x^2\delta_y^2\left(\Delta t\,\frac{\partial u}{\partial t}\Big|_{j,k}\right)$$

$$\approx \frac{\Delta t^3}{\Delta x^2}\delta_x^2\frac{\delta_y^2}{\Delta y^2}\left(\Delta t\,\frac{\partial u}{\partial t}\Big|_{j,k}\right)$$

$\because \frac{\delta_y^2}{\Delta y^2} \approx \frac{\partial^2}{\partial y^2}$ $\quad \approx \frac{\Delta t^3}{\Delta x^2}\delta_x^2\left(\left(\frac{\partial u}{\partial t}\right)_{yy} + O(\Delta y^2)\right)$

$\because \frac{\delta_x^2}{\Delta x^2} \approx \frac{\partial^2}{\partial x^2}$ $\quad \approx \Delta t^3 \frac{\partial^5 u}{\partial t\,\partial x^2\,\partial y^2}$

$$\therefore \frac{\Delta t^2}{\Delta x^2\Delta y^2}\delta_x^2\delta_y^2\Delta u_{j,k}^n = O(\Delta t^3)$$

$$I\Delta u_{j,k}^n = O(\Delta t).$$

implicit Euler: ~~The truncation error is $O(\Delta t)+O(\Delta x^2)+O(\Delta y^2)$~~

$$\left(I-\frac{\Delta t}{\Delta x^2}\delta_x^2-\frac{\Delta t}{\Delta y^2}\delta_y^2\right)\Delta u_{j,k}^n = \Delta t\left(\frac{1}{\Delta x^2}\delta_x^2+\frac{1}{\Delta y^2}\delta_y^2\right)u_{j,k}^n$$

$\|$

$O(\Delta t)$ $\quad$ $O(\Delta t)$

$$\left(I-\frac{\Delta t}{\Delta x^2}\delta_x^2\right)\left(I-\frac{\Delta t}{\Delta y^2}\delta_y^2\right)\Delta u_{j,k}^n = \Delta t\left(\frac{1}{\Delta x^2}\delta_x^2+\frac{1}{\Delta y^2}\delta_y^2\right)u_{j,k}^n + \underbrace{\frac{\Delta t^2}{\Delta x^2\Delta y^2}\delta_x\delta_y^2\Delta u_{j,k}^n}$$

~~$O(\Delta t^3)$ drop it~~

drop the truncancation error is still same as before. ∴ drop 有
有什么关系.

approximate implicit Euler

$$\left(I-\frac{\Delta t}{\Delta x^2}\delta_x^2\right)\left(I-\frac{\Delta t}{\Delta y^2}\delta_y^2\right)\Delta u_{j,k}^n = \Delta t\left(\frac{1}{\Delta x^2}\delta_x^2+\frac{1}{\Delta y^2}\delta_y^2\right)u_{j,k}^n$$

~~For θ method~~

$$\left(I-\frac{\theta\Delta t}{\Delta x^2}\delta_x^2\right)\left(I-\frac{\theta\Delta t}{\Delta y^2}\delta_y^2\right)\Delta u_{j,k}^n = \Delta t\left(\frac{1}{\Delta x^2}\delta_x^2+\frac{1}{\Delta y^2}\delta_y^2\right)u_{j,k}^n$$

How to solve the equation.

let $U_{j,k} = \left(I-\frac{\theta\Delta t}{\Delta y^2}\delta_y^2\right)\Delta u_{j,k}^n$

$$\left[I-\frac{\theta\Delta t}{\Delta x^2}\delta_x^2\right]U_{j,k} = \underbrace{\Delta t\left(\frac{1}{\Delta x^2}\delta_x^2+\frac{1}{\Delta y^2}\delta_y^2\right)u_{j,k}^n}_{d_{j,k}}$$

(先 solve $U_{j,k}$, then solve $\Delta u_{j,k}^n$ i.e. $u_{j,k}^{n+1}$)

ADI ①

Implementation of ADI :

$$\begin{cases} u_t = \Delta u + f(x,y,t) & \text{in } \Omega \\ u|_{\partial\Omega} = g(x,y,t), & (x,y) \text{ on } \partial\Omega \end{cases}$$

The Crank-Nicolson scheme.

Time symmetric. ADI (AF) method :

$$\left(I - \frac{V_x}{2}\delta_x^2\right)\left(I - \frac{V_y}{2}\delta_y^2\right) u_{j,k}^{n+1} = \left(I + \frac{V_x}{2}\delta_x^2\right)\left(I + \frac{V_y}{2}\delta_y^2\right) u_{j,k}^n + \Delta t f(x_j, y_k, t_n + \tfrac{1}{2}\Delta t)$$

$$V_x = \frac{\Delta t}{\Delta x^2}$$

$$V_y = \frac{\Delta t}{\Delta y^2}$$

Define : $V_{j,k} = \left(I - \frac{V_y}{2}\delta_y^2\right) u_{j,k}^{n+1}$

$$\begin{cases} \left(I - \frac{V_x}{2}\delta_x^2\right) V_{j,k} = \left(I + \frac{V_x}{2}\delta_x^2\right)\left(I + \frac{V_y}{2}\delta_y^2\right) u_{j,k}^n + \Delta t f(x_j, y_k, t_n + \tfrac{1}{2}\Delta t) \\ \left.\begin{aligned} V_{0,k} &= \left(I - \frac{V_y}{2}\delta_y^2\right) u_{0,k}^{n+1} \\ V_{J,k} &= \left(I - \frac{V_y}{2}\delta_y^2\right) u_{J,k}^{n+1} \end{aligned}\right\} \text{bndary condition} \end{cases}$$

Given J and k and N (how many time steps), T, $u^0(x_j, y_k)$

Initial calculation : Δx, Δy, Δt, V_x, V_y, $t=0$

and make mesh (x_j, y_k).

x_j, $j = 0 \cdots J$

y_k, $k = 0 \cdots k$

$u_{j,k} = u^0(x_j, y_k)$ $\quad j = 0, 1 \cdots J$, $k = 0, 1 \cdots k$

Define work arrays. : ax[0:J], bx, cx, dx

ay[0:k], by, cy, dy

$V_{j,k}^{(n)}$, $W_{j,k}^{(n+1)}$

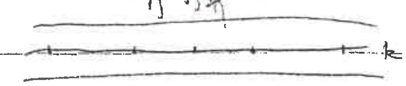
For $n = 0 \cdots N-1$: (time loop. usually called time integration)

1° calculate BCs for u^{n+1}

$W_{j,k} = g(x_j, y_k, t + \Delta t)$ for $j = 0$ or $j = J$ or $k = 0$ or $k = K$

上式即为 $\frac{-\theta\Delta t}{\Delta x^2}V_{j-1,k} + (1+\frac{\theta\Delta t}{\Delta x^2})\quad V_{j,k} - \frac{\theta\Delta t}{\Delta x^2}V_{j+1,k} = d_{j,k}$

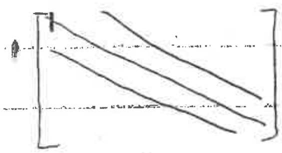
行-行解



For $k=1,2,\cdots K-1$

$$\begin{cases} -\frac{\theta\Delta t}{\Delta x^2}V_{j-1,k} + (1+\frac{\theta\Delta t}{\Delta x^2})V_{j,k} - \frac{\theta\Delta t}{\Delta x^2}V_{j+1,k} = d_{j,k} \qquad j=1\cdots J-1 \\ V_{0,k} = (I - \frac{\theta\Delta t}{\Delta y^2}\delta_y^2)\Delta U^n_{0,k} = \frac{-\theta\Delta t}{\Delta y^2}\Delta U^n_{0,k-1} + (1+\frac{2\theta\Delta t}{\Delta y^2})\Delta U^n_{0,k} - \frac{\theta\Delta t}{\Delta y^2}\Delta U^n_{0,k+1} \\ V_{J,k} = (I - \frac{\theta\Delta t}{\Delta y^2}\delta_y^2)\Delta U^n_{J,k} = \end{cases}$$

the matrix looks like this:



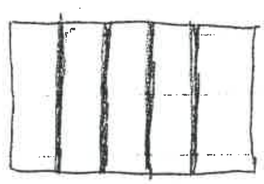
$$(J+1)\times(J+1) \quad \begin{bmatrix} V_{0k} \\ \vdots \\ V_{J,k} \end{bmatrix} = \begin{bmatrix} 0 \\ d_{1,k} \\ \vdots \\ d_{J-1,k} \\ 0 \end{bmatrix}$$

x-sweep

for $\Delta U^n_{j,k}$

For $j=1\cdots J-1$

$$\begin{cases} -\frac{\theta\Delta t}{\Delta y^2}\Delta U^n_{j,k-1} + (1+\frac{2\theta\Delta t}{\Delta y^2})\Delta U^n_{j,k} - \frac{\theta\Delta t}{\Delta y^2}\Delta U^n_{j,k+1} = \underset{\text{just need inside}}{\underline{V_{j,k}}} \qquad k=1\cdots K-1, \\ \left.\begin{array}{l}\Delta U^n_{j,0} = ? \\ \Delta U^n_{j,K} = ?\end{array}\right\} \text{given boundary condition} \end{cases}$$



y-sweep

Ex: Show the stability for the new scheme. (*)

something like θ-method

Lecture 15

3月3日

$Au = b$

$\uparrow$ result from the discretization of a PDE

$$A \approx \underbrace{\left(I - \frac{\theta \Delta t}{\Delta x^2}\delta_x^2\right)\left(I - \frac{\theta \Delta t}{\Delta x^2}\delta_y^2\right)}_{P}$$

$A - P = O(\Delta t^2)$.

$(P^{-1}A)u = P^{-1}b$ easy to solve using iterative methods.

speed $\propto$ $k(A) = \|A\| \cdot \|A^{-1}\|$

$\wr$

$k(A) = O(\frac{1}{h^2})$

$$\frac{\|v^{(i+1)} - v^*\|}{\|v^{(i)} - v^*\|} = \frac{\sqrt{k(A)} - 1}{\sqrt{kA} + 1}$$

$\rightarrow 1$ 的话则 趋于 $\rightarrow 0$

$\sim 1 - O(h)$.

但P很难解 ∴要用特殊的方法来解.

Apply the θ-method and operator splitting to

$u_t = u_{xx} + u_{yy}$

$$\Rightarrow \left(I - \frac{\Delta t}{\Delta x^2}\delta_x^2\right)\left(I - \frac{\Delta t}{\Delta y^2}\delta_y^2\right)\Delta u^n_{j,k} = \Delta t\left[\frac{1}{\Delta x^2}\delta_x^2 + \frac{1}{\Delta y^2}\delta_y^2\right]u^n_{j,k}$$

For purpose of stability analysis,

$\Delta u^n_{j,k} = u^n_{j,k} - u^n_{j,k}$ then get the system like this:

$$\left(I - \frac{\theta \Delta t}{\Delta x^2}\delta_x^2\right)\left(I - \frac{\theta \Delta t}{\Delta y^2}\delta_y^2\right)u^{n+1}_{j,k} = \Delta t\left(\frac{1}{\Delta x^2}\delta_x^2 + \frac{1}{\Delta y^2}\delta_y^2\right)u^n_{j,k} + \left(I - \frac{\theta\Delta t}{\Delta x^2}\delta_x^2\right)\left(I - \frac{\theta\Delta t}{\Delta y^2}\delta_y^2\right)u^n_{j,k}$$

$$= \boxed{\left[I + \frac{(1-\theta)\Delta t}{\Delta x^2}\delta_x^2 + \frac{(1-\theta)\Delta t}{\Delta y^2}\delta_y^2 + \frac{\theta^2\Delta t^2}{\Delta x^2 \Delta y^2}\delta_x^2\delta_y^2\right]u^n_{j,k}}$$

$\rightarrow$ 中间步骤

$$= \left(I + \frac{(1-\theta)\Delta t}{\Delta x^2}\delta_x^2\right)\left(I + \frac{(1-\theta)\Delta t}{\Delta y^2}\delta_y^2\right)u^n_{j,k}$$

$$+ \underbrace{\left[-\frac{(1-\theta)^2\Delta t^2}{\Delta x^2\Delta y^2}\delta_x^2\delta_y^2 + \frac{\theta^2\Delta t^2}{\Delta x^2\Delta y^2}\delta_x^2\delta_y^2\right]}_{= (\theta^2 - (1-\theta)^2)\Delta t^2/\Delta x^2\Delta y^2 \cdot \delta_x^2\delta_y^2} u^n_{j,k} \qquad (*)$$

For the stability analysis:

$$u_{j,k}^n = \lambda^n e^{im\pi\Delta x j + i\ell\pi\Delta y k} \qquad (m,\ell)$$

$$\delta_x^2 u_{j,k}^n = -4\sin^2\frac{m\pi\Delta x}{2}\, u_{j,k}^n$$

$$\delta_y^2 u_{j,k}^n = -4\sin^2\frac{\ell\pi\Delta y}{2}\, u_{j,k}^n$$

so the L.h.s. of $(*)$ $= \left(1+\frac{\theta\Delta t}{\Delta x^2}4\sin^2\frac{m\pi\Delta x}{2}\right)\left(1+\frac{\theta\Delta t}{\Delta y^2}4\sin^2\frac{\ell\pi\Delta y}{2}\right)u_{j,k}^{n+1}$

$$= \left(1-\frac{(1-\theta)\Delta t}{\Delta x^2}4\sin^2\frac{m\pi\Delta x}{2}\right)\left(1-\frac{(1-\theta)\Delta t}{\Delta y^2}4\sin^2\frac{\ell\pi\Delta y}{2}\right)u_{j,k}^n$$

$$+\left(\theta^2-(1-\theta)^2\right)\frac{\Delta t^2}{\Delta x^2\Delta y^2}16\sin^2\frac{m\pi\Delta x}{2}\sin^2\frac{\ell\pi\Delta y}{2}\,u_{j,k}^n$$

let $\xi = \sin^2\left(\frac{m\pi\Delta x}{2}\right)$ $\quad \eta = \sin^2\left(\frac{\ell\pi\Delta y}{2}\right)$ $\quad 0\le \xi,\eta\le 1$

$$\left(1+\frac{4\theta\Delta t}{\Delta x^2}\xi\right)\left(1+\frac{4\theta\Delta t}{\Delta y^2}\eta\right)\lambda = \left(1-\frac{4(1-\theta)\Delta t}{\Delta x^2}\xi\right)\left(1-\frac{4(1-\theta)\Delta t}{\Delta y^2}\eta\right)$$

$$+\left[\theta^2-(1-\theta)^2\right]16\frac{\Delta t^2\xi\eta}{\Delta x^2\Delta y^2}$$

$|\lambda|\le 1$ then stable

$$\theta=\frac{1}{2} \qquad \lambda = \frac{\left(1-\frac{2\Delta t}{\Delta x^2}\xi\right)\left(1-\frac{2\Delta t}{\Delta y^2}\eta\right)}{\left(1+\frac{2\Delta t}{\Delta x^2}\xi\right)\left(1+\frac{2\Delta t}{\Delta y^2}\eta\right)}$$

$|\lambda|\le 1$ (如果 $\frac{2\Delta t}{\Delta x^2}\xi > 1$ 则写成 $\left(\frac{2\Delta t}{\Delta x^2}\xi-1\right)\left(\frac{2\Delta t}{\Delta y^2}\eta-1\right)$ 这样 easy to see)

$\theta = 0$

time symmetry: → indicate how fast the (m,ℓ) mode is damped.

$$u_t = u_{xx}+u_{yy}$$

$$u(x,y,z) = \sum_{m,\ell} u_{m,\ell}\, e^{-\underbrace{[(m^2+\ell^2)\pi^2]}_{\text{damping factor}}t}\, \underbrace{e^{im\pi x + i\ell\pi y}}_{(m,\ell)\text{ fourier mode}}$$

$\left\{\begin{array}{l}\text{proper damping factor,}\\ \text{numerical diffusion.}\end{array}\right.$

similar to transformation $t \to -t$.