Hour 20 Lecture 17

Chapter 4 Hyperbolic Equations in 1 D

§4.1 Characteristics

0°. Introduction $u_t + a u_x = \frac{1}{Re} u_{xx}$ (convection, diffusion)

1. $u_t + a u_x = 0$

   $u = u(x - at)$

   characteristics

2. $u_t = u_{xx}$

3. systems $u = \begin{bmatrix} u^1(x,t) \\ u^2(x,t) \\ \vdots \\ u^N(x,t) \end{bmatrix}$, $f = \begin{bmatrix} f^1(u) \\ \vdots \\ f^N(u) \end{bmatrix}$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$ Conservation Laws

$$A(u) = \frac{\partial f}{\partial u}$$

$$\frac{\partial f(u)}{\partial x} = A \frac{\partial u}{\partial x}$$

row vector v_1

$$v_1 A = \lambda_1 v_1$$

$$\vdots$$

$$v_N A = \lambda_N v_N$$

$$S = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}$$

$\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$

$$SA = \Lambda S$$

$$S \frac{\partial u}{\partial t} + \Lambda S \frac{\partial u}{\partial x} = 0$$

r: $\frac{\partial r}{\partial t} = S \frac{\partial u}{\partial t}$.

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§§4.2–4.4 Upwind Schemes

1. Consider $U_t + aU_x = 0 \quad a>0$

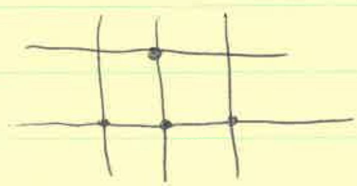
$$\frac{u_j^{n+1}-u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

$U_j^n = (\lambda)^n e^{ikx_j} \qquad x_j = j\Delta x$

$\lambda (= \lambda(k)) = 1 - \nu i \sin k\Delta x$ | central scheme

$\nu = \frac{a\Delta t}{\Delta x}$

$|\lambda| > 1$ for all k except $\sin k\Delta x = n\pi$.



2. Upwind Scheme

a) design

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} = \frac{1}{Re}\Delta\vec{u}.$$

$$\begin{cases} U_t + aU_x = 0 & a>0 \text{ constant} \\ U_t + aU_x = 0 & a<0 \text{ constant} \end{cases}$$

$U_t + aU_x = 0 \qquad a = a(x,t,u)$

why central scheme does not work?

• Give the scheme here.

b) Analysis: Stability

$U_t + aU_x = 0 \qquad a>0$

① Fourier Analysis:

$$U_j^n = \lambda^n e^{ikx_j}$$

$$\lambda = \lambda(k) = 1 - \nu(1 - e^{-ik\Delta x})$$

$$|\lambda| = 1 - 4\nu(1-\nu)\sin^2\tfrac{1}{2}k\Delta x \Rightarrow \nu < 1$$

② CFL Condition (illustration of CFL condition)

- domain of dependente
- CFL condition is not suficient !
  (central scheme)

c) Analysis: Convergence

$$O(\Delta t, \Delta x) \quad \text{if } \nu < 1$$

d) Analysis: Phase and amplitude errors

$$\arg\lambda = -\tan^{-1}\left[\frac{\nu\sin k\Delta x}{(1-\nu)+\nu\cos k\Delta x}\right]$$

Let $\xi = k\Delta x$

$$\arg\lambda \approx -\nu\xi\left[1 - \tfrac{1}{6}(1-\nu)(1-2\nu)\xi^2 + \cdots\right]$$

physical meaning

Exact: $u(x,t) = e^{i(kx+\omega t)}$ $\quad \omega = -ak$

phase change in Δt $= -ak\Delta t = -\nu k\Delta x$