Hour 23, 24 Lecture 18 ~~Lecture 19~~

§§4.5–4.6 The Lax-Wendroff scheme

1. Consider

$$u_t + a(x,t)\,u_x = 0$$

$$u_t \sim \frac{U_j^{n+1} - U_j^n}{\Delta t}$$

$$u(x, t+\Delta t) = u(x,t) + \Delta t\, u_t(x,t) + \tfrac{1}{2}(\Delta t)^2 u_{tt}(x,t) + O(\Delta t^3)$$

$$u_t = -a u_x$$

$$u_{tt} = -a_t u_x - a u_{xt} = -a_t u_x + a(a u_x)_x$$

$$u(x,t+\Delta t) = u(x,t) - \Delta t\, a u_x + \frac{\Delta t^2}{2}\left[-a_t u_x + a(a u_x)_x\right] + O(\Delta t^3)$$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = -a_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} + \frac{\Delta t}{2}\left\{-(a_t)_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} + \frac{a_j^n}{\Delta x}\left[a_{j+\frac12}\frac{U_{j+1}^n - U_j^n}{\Delta x} - a_{j-\frac12}\frac{U_j^n - U_{j-1}^n}{\Delta x}\right]\right\}$$

$$O(\Delta t^2 + \Delta x^2)$$

a constant:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = -a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} + \frac{\Delta t\, a}{2\Delta x}\left[\cdots\right]$$

$$\sim\ u_t + a u_x = \frac{\Delta t}{2} a^2 u_{xx}$$

$$u_t + a u_x = \frac{\Delta t\, a}{2}(a u_x)_x \qquad (a_t = 0)$$

Questions

- truncation error $O(\Delta t^2 + \Delta x^2)$
- convergence $O(\Delta t^2 + \Delta x^2)$ if u is sufficiently smo
  ? if u is non-smooth.
- stability (CFL)

$$\frac{a \Delta t}{\Delta x} \le 1$$

- phase and amplitude errors

$$\lambda(k) = 1 - 2\nu^2 \sin^2 \tfrac{1}{2}k\Delta x - i\nu \sin k\Delta x$$

$$|\lambda(k)| = \sqrt{1 - 4\nu^2(1-\nu^2)\sin^4 \tfrac{1}{2}k\Delta x}$$

damping but of order $\xi^4 = (k\Delta x)^4$

$$\arg \lambda(k) = -\tan^{-1}\left[\frac{\nu \sin k\Delta x}{1 - 2\nu^2 \sin^2 \frac{1}{2}k\Delta x}\right]$$

$$\sim -\nu\xi\left[1 - \tfrac{1}{6}(1-\nu^2)\xi^2 + \cdots\right]$$

of order $\xi^2 = (k\Delta x)^2$

- monotonicity (NOT!)

Example

$$\begin{cases} u_t + a(x,t)\, u_x = 0 \\ u(x,0) = \begin{cases} 1 & \text{if } 0.2 \le x \le 0.4 \\ 0 & \text{otherwise} \end{cases} \\ u(0,t) = 0 \end{cases}$$

$$a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4}$$

$\Delta t = \Delta x$. $\quad O((\Delta x)^{2/3})$

Figure 4.7.

2. For conservation laws

$$u_t + (f(u))_x = 0$$

$$u(t+\Delta t) = u(t) + \Delta t\, u_t + \frac{\Delta t^2}{2!} u_{tt} + O(\Delta t^3)$$

$$u_t = -f_x$$

$$u_{tt} = -f_{xt} = -(f_t)_x = -(f_u u_t)_x$$
$$= +(f_u f_x)_x$$

$$a(u) = f_u$$

$$u(t+\Delta t) = u(t) - \Delta t\, f_x + \frac{\Delta t^2}{2}(a f_x)_x$$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = -\frac{f(U_{j+1}^n) - f(U_{j-1}^n)}{2\Delta x} + \frac{\Delta t}{2\Delta x}\left[a_{j+1/2}^n \frac{f(U_{j+1}^n) - f(U_j^n)}{\Delta x} - a_{j-1/2}^n \cdots \right]$$

$$a^n_{j+1/2} = a(U^n_{j+1/2}) = a\left(\frac{U^n_{j+1}+U^n_j}{2}\right)$$

$$u_t + f_x = \frac{\Delta t}{2}(a f_x)_x$$

$$u_t + f_x = \frac{\Delta t}{2}(f_u^2 u_x)_x$$

$\Uparrow$ artificial viscousity

Example Burgers' Equation

$$u_t + u u_x = 0$$

$$u_t + \tfrac{1}{2}(u^2)_x = 0$$

$$u(x,0) = e^{-10(4x-1)^2}$$

Figure 4.10

Variant of the Lax-Wendroff scheme

$$U^{n+1/2}_{j+1/2} = \frac{1}{2}\left[U^n_j + U^n_{j+1}\right] - \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)\left[f(U^n_{j+1}) - f(U^n_j)\right]$$

$$U^{n+1}_j = U^n_j - \frac{\Delta t}{\Delta x}\left[f(U^{n+1/2}_{j+1/2}) - f(U^{n+1/2}_{j-1/2})\right]$$

Readings §§ 4.7, 4.8, 4.9
Box Scheme, Leap-frog scheme
comparison of phase and amplitude errors.