Lecture 1

HW 1 (a)
Prob 1.1 (§1.1)

Chapter 1 Introduction to FEM for elliptic problems

§1.1 Variational formulation of a one-dimensional problem

- introduce notation: V, $(\cdot,\cdot)$, and $F(\cdot)$
- introduce weak (variational) formulations
- prove that $(D) \Rightarrow (V) \Leftrightarrow (M)$
  $\left(\Leftarrow \text{ if } u \in C^2(\Omega)\right)$

• Consider the boundary value problem (BVP)

$$(D) \quad \begin{cases} -u''(x) = f(x) & x \in \Omega = (0,1) \\ u(0)=0, \; u(1)=0 \end{cases}$$

where f is a given continuous function.
(D): Stands for Differential ~~Equation~~ form
(Strong form)

• To introduce weak (or variational or integral) formulations, we introduce some notation.

linear function space

$$V = \left\{ v \;\middle|\; \begin{array}{l} \cdot\ v \text{ is continuous function on } [0,1] \\ \cdot\ v' \text{ is piesewise continuous, bounded on } (0,1) \\ \cdot\ v(0) = v(1) = 0 \end{array} \right\}$$

inner product:

$$(v, w) = \int_0^1 v(x)\,w(x)\,dx$$

- Two weak (variational or integral) formulations.

(V): (Galerkin or variational or weak formulation).

find $u \in \bar{V}$ such that

$$(u', v') = (f, v) \quad \forall v \in \bar{V}$$

(M) (minimization or Ritz formulation)

find $u \in \bar{V}$ such that

$$F(u) \leq F(v) \quad \forall v \in \bar{V}$$

where

$$F(v) = \tfrac{1}{2}(v', v') - (f, v)$$

Theorem

$$(D) \underset{\left(\Leftarrow \text{ if } u \in C^2(\Omega)\right)}{\Rightarrow} (V) \Longleftrightarrow (M)$$

Proof (1) $(D) \Rightarrow (V)$

Assume that u is a solution of (D). We want to show that

$$(u', v') = (f, v) \quad \forall v \in \bar{V}.$$

For any $v \in V$, we multiply (D) by v and integrate over Ω and have

$$-\int_0^1 u'' v \, dx = \int_0^1 f v \, dx$$

Integrating by parts, we get

$$\underbrace{-u'v\Big|_0^1}_{=0 \text{ due to BCs}} + \int_0^1 u'v' \, dx = \int_0^1 f v \, dx$$

$$\Rightarrow \qquad (u', v') = (f, v)$$

Since v is arbitrary in V, so we have (V).

(2) (V) $\Rightarrow$ (M)

Assume that u is a solution of (V). We want to show $F(u) \leq F(v)$ $\forall v \in V$.

Let $v = u + w$ or $w = v - u$

Goal $\Rightarrow$ $F(u+w) \geq F(u)$ $\forall w \in V$

$$F(u+w) = \frac{1}{2}\big((u+w)', (u+w)'\big) - (f, u+w)$$

$$= \frac{1}{2}(u', u') + (u', w') + \frac{1}{2}(w', w') - (f, u) - (f, w)$$

$$= F(u) + \underbrace{\big[(u', w') - (f, w)\big]}_{=0} + \underbrace{\frac{1}{2}(w', w')}_{\geq 0}$$

$$\geq F(u).$$

Thus (V) $\Rightarrow$ (M).

(3) (M) $\Rightarrow$ (V)

Assume that u is a solution of (M). We want to show that

$$(u', v') = (f, v) \quad \forall v \in V.$$

Since u is a soln of (M):

$$F(v) \geq F(u) \qquad \forall v \in V.$$

or

$$F(u + \varepsilon v) \geq F(u)$$

for any $v \in V$ and $\varepsilon \in \mathbb{R}$

Define

$$g(\varepsilon) = F(u + \varepsilon v)$$

Then $g(\varepsilon)$ has a minimum value @ $\varepsilon = 0$

or $g'(0) = 0$

$$g(\varepsilon) = \cancel{\tfrac{1}{2}} F(u + \varepsilon v) = \frac{1}{2}\int_0^1 \left((u + \varepsilon v)'\right)^2 dx - \int_0^1 f\,(u + \varepsilon v)\,dx$$

$$g'(0) = \int_0^1 u'v'\,dx - \int_0^1 f v\,dx$$

$$= (u', v') - (f, v)$$

$g'(0) = 0 \Rightarrow (u', v') = (f, v)$

Since v is arbitrary in V so we have (V).

(4) $(V)\ u \in C^2(\Omega) \Rightarrow (D)$

Assume that u is a solution of (V) and u'' is continuous on $(0,1)$

Then $(u', v') = (f, v) \quad \forall v \in V$

or $$\int_0^1 u'v'\,dx - \int_0^1 fv\,dx = 0$$

$$\int_0^1 (\underbrace{-u'' - f}_{\text{continuous}})\, v\,dx = 0 \qquad \forall v \in V$$

By the vanishing theorem $\Rightarrow$

$$-u'' - f = 0$$

or $$-u'' = f. \qquad \#$$

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