Lecture 2

§1.2 FEM with piecewise linear functions

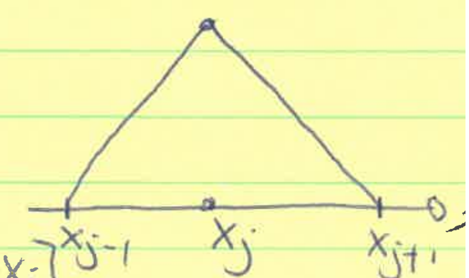
- Mesh (triangulation, grid, partition)

$$x_0 = 0 < x_1 < \cdots < x_M < x_{M+1} = 1$$

(special example: uniform mesh
$$x_j = jh, \quad h = \frac{1}{M+1}, \quad j = 0, 1, \cdots, M+1)$$

Denote
$$I_j = [x_{j-1}, x_j]$$
$$h_j = x_j - x_{j-1} \qquad j = 1, \cdots, M+1$$
$$h = \max_j h_j$$

- linear basis functions



$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h_j} & x \in [x_{j-1}, x_j] \\ \dfrac{x_{j+1} - x}{h_{j+1}} & x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise} \end{cases}$$

Linear FE Space
$$V^h = \text{span}\{\varphi_1(x), \cdots, \varphi_M(x)\}$$
$$= \left\{ v(x) \,\middle|\, v(x) = \sum_{j=1}^{M} \eta_j \varphi_j(x), \ \eta_j \in \mathbb{R} \right\}$$

Note: $V^h \subset V$

- Linear FE approximation

Find $u_h \in V^h$ such that

$$\left.\begin{aligned}(u_h', v_h') = (f, v_h) \quad \forall v_h \in V^h\end{aligned}\right\} (V_h)$$

- Matrix form

$$u_h = \sum_{j=1}^{M} \xi_j \varphi_j(x)$$

$$\sum_{j=1}^{M} \xi_j (\varphi_j', v_h') = (f, v_h) \quad \forall v_h \in V^h$$

take $v_h = \varphi_i$ $(i = 1, \cdots, M)$

$$\sum_{j=1}^{M} (\varphi_j', \varphi_i') \xi_j = (f, \varphi_i) \quad i = 1, \cdots, M$$

$$\Rightarrow \qquad A\xi = b$$

where $A_{ij} = (\varphi_j', \varphi_i')$

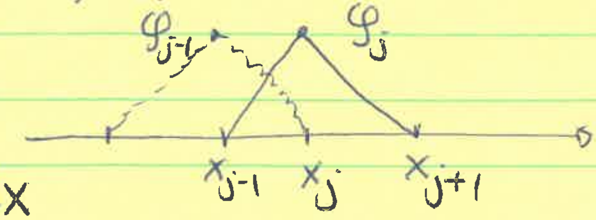
$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_M \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix}$$

$$b_i = (f, \varphi_i)$$

A: stiffness matrix

- Properties of A:
  - symmetric: $A_{ij} = (\varphi_j', \varphi_i') = (\varphi_i', \varphi_j') = A_{ji}$
  - sparsity

$$a_{ij} = 0 \qquad j \neq i-1, i, i+1$$



$$a_{i,i-1} = (\varphi_{i-1}', \varphi_i')$$

$$= \int_0^1 \varphi_{i-1}' \varphi_i' \, dx$$

$$= \int_{x_{i-1}}^{x_i} \frac{-1}{h_i} \frac{+1}{h_i} \, dx = -\frac{1}{h_i}$$

$$a_{i,i+1} = a_{i+1,i} = -\frac{1}{h_{i+1}}$$

$$a_{i,i} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_i}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{1}{h_{i+1}}\right)^2 dx$$

$$= \frac{1}{h_i} + \frac{1}{h_{i+1}}$$

$$A = \begin{bmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \frac{1}{h_1} + \frac{1}{h_2} & -\frac{1}{h_2} & & \\ -\frac{1}{h_2} & \frac{1}{h_2} + \frac{1}{h_3} & -\frac{1}{h_3} & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

  - SPD

$$\xi^T A \xi = \sum_{ij} \xi_i A_{ij} \xi_j = \sum_{ij} (\varphi_i', \varphi_j') \xi_i \xi_j$$

$$= \left(\sum_i \xi_i \varphi_i', \sum_j \xi_j \varphi_j' \right)$$

$$= (u_h', u_h')$$

$$= \int_0^1 (u_h')^2 \, dx$$

$$\geq C \int_0^1 (u_h')^2 dx \quad \text{(Poincaré's inequality)}$$

$$> 0 \quad \text{when } \xi \neq 0.$$

- RHS.

$$b_i = (f, \varphi_i)$$
$$= \int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)\,dx + \int_{x_i}^{x_{i+1}} f(x)\varphi_i(x)\,dx$$

numerical integration or quadrature.

For example: 2-point Gaussian quadrature rule

$$\int_{-1}^{1} f(x)\,dx \approx f\left(\tfrac{1}{\sqrt{3}}\right) + f\left(-\tfrac{1}{\sqrt{3}}\right)$$

$$\int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)\,dx \approx h_i f\left(x_{i-1} + \frac{1+\frac{1}{\sqrt{3}}}{2} h_i\right)\varphi_i(\cdots)$$
$$+ h_i f\left(x_{i-1} + \frac{1-\frac{1}{\sqrt{3}}}{2} h_i\right)\varphi_i(\cdots)$$

- Assembling of A and b (element-by-element)

— Set $A = 0$ $b = 0$ (initialization)

— for $i = 1, \cdots, M+1$ (on element $I_i = [x_{i-1}, x_i]$)

$$\begin{cases} a_{ii} = a_{ii} + (\varphi_i', \varphi_i')_{I_i} = a_{ii} + \int_{x_{i-1}}^{x_i} \varphi_i' \varphi_i'\,dx \\ a_{i-1,i-1} = a_{i-1,i-1} + (\varphi_{i-1}', \varphi_{i-1}')_{I_i} \\ a_{i,i-1} = a_{i,i-1} + (\varphi_{i-1}', \varphi_i')_{I_i} \\ a_{i-1,i} = a_{i-1,i} + (\varphi_i', \varphi_{i-1}')_{I_i} \end{cases}$$

$$\begin{cases} b_i = b_i + (f, \varphi_i)_{I_i} \\ b_{i-1} = b_{i-1} + (f, \varphi_{i-1})_{I_i} \end{cases}$$

(It seems that all computation is done on element I_i).

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