Lecture 2

§2.5 Difference Notation and Truncation Error

- **Finite Difference Notation:**

Consider a function $u(x,t)$ of x and t.

Forward differences:

$$\Delta_{+x} u(x,t) := u(x+\Delta x, t) - u(x,t)$$
$$\Delta_{+t} u(x,t) := u(x, t+\Delta t) - u(x,t)$$

Backward differences:

$$\Delta_{-x} u(x,t) := u(x,t) - u(x-\Delta t, t)$$
$$\Delta_{-t} u(x,t) := u(x,t) - u(x, t-\Delta t)$$

central differences:

$$\delta_t u(x,t) := u(x, t+\tfrac{1}{2}\Delta t) - u(x, t-\tfrac{1}{2}\Delta t)$$
$$\delta_x u(x,t) := u(x+\tfrac{1}{2}\Delta x, t) - u(x-\tfrac{1}{2}\Delta x, t)$$

central differences (double interval)

$$\Delta_{0x} u(x,t) := \frac{1}{2}(\Delta_{+x} + \Delta_{-x}) u(x,t)$$
$$= \frac{1}{2}\left[u(x+\Delta x, t) - u(x-\Delta x, t)\right]$$

Example 1. $\delta_x^2 u(x,t) = \Delta_{+x}\Delta_{-x} u(x,t) = \Delta_{-x}\Delta_{+x} u(x,t)$

- **Big O and Small O:**

Assume that we are given two sequences $\{\alpha_i\}$ and $\{\beta_i\}$: $\alpha_i \to 0$, $\beta_i \to 0$ as $i \to +\infty$.
if there exists a constant K such that

$$|\alpha_i| \le K |\beta_i| \quad \text{as } i \to +\infty$$

we say $\alpha_i = O(\beta_i)$ as $i \to +\infty$.

(same order)

If $|\alpha_i|/|\beta_i| \to 0$ as $i \to \infty$,
we say
$\alpha_i = o(\beta_i)$ as $i \to +\infty$.
(higher order)

- Taylor Series

$\boxed{\text{Theorem}}$ If a function $u(x)$ has derivatives of orders up to $n+1$ in some interval I containing a given point x_0, then u can be expended as

$$u(x) = u(x_0) + u'(x_0)(x-x_0) + \cdots + \frac{u^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{u^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

for $x \in I$

where ξ is a point between x_0 and x.

If u has derivatives of all orders, then

$$u(x) = u(x_0) + u'(x_0)(x-x_0) + \frac{u''(x_0)}{2!}(x-x_0)^2 + \cdots$$
$$= \sum_{i=0}^{\infty} \frac{(x-x_0)^i}{i!} u^{(i)}(x_0).$$

Example 2 Assume that $u(x,t)$ has the infinite number of derivatives. w.r.t. x and t. Then we have

$$\Delta_{+t} u(x,t) = u_t \Delta t + \frac{1}{2} u_{tt} (\Delta t)^2 + \frac{1}{6} u_{ttt} (\Delta t)^3 + \cdots$$
$$= \cdots\cdots + O(\Delta t^3)$$
$$= u_t \Delta t + \frac{1}{2} u_{tt}(x,\eta) \Delta t^2$$

$$\delta_x^2 u(x,t) = u_{xx} \Delta x^2 + \frac{1}{12} u^{(iv)} \Delta x^4 + \cdots$$
$$= u_{xx} \Delta x^2 + \frac{1}{12} u^{(iv)} \Delta x^4 + O(\Delta x^6)$$
$$= u_{xx} \Delta x^2 + \frac{1}{12} u^{(iv)}(\xi, t) \Delta x^4$$

Truncation Error:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

can be denoted as

$$Lu := \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

Where Operator $L := \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$.

Explicit Scheme:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

or $$\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} = 0$$

or $$\frac{\Delta_{+t} U_j^n}{\Delta t} - \frac{\delta_x^2 U_j^n}{\Delta x^2} = 0$$

Denote: $$L_{FD} U_j^n := \frac{\Delta_{+t} U_j^n}{\Delta t} - \frac{\delta_x^2 U_j^n}{\Delta x^2} = 0$$

DE:	$Lu(x,t) = 0$	$0 < x < 1, \quad 0 < t$
FDE:	$L_{FD} U_j^n = 0$	$1 < j < J-1, \quad 1 \le n$

$$T(x,t) := L_{FD}\, u(x,t) = \frac{\Delta_{+t} u(x,t)}{\Delta t} - \frac{\delta_x^2 u(x,t)}{\Delta x^2}$$

$u(x,t)$ exact solution of the PDE.

$$T(x,t) = (u_t - u_{xx}) + \tfrac{1}{2}u_{tt}\Delta t - \tfrac{1}{12}u_{xxxx}(\Delta x)^2 + \cdots$$

$$= Lu + \tfrac{1}{2}u_{tt}\Delta t - \tfrac{1}{12}u_{xxxx}\Delta x^2 + O(\Delta t^2, \Delta x^4)$$

$$= Lu + \tfrac{1}{2}u_{tt}(x,\eta)\Delta t - \tfrac{1}{12}u_x^{(iv)}(\xi,t)\Delta x^2$$

$$|T(x,t)| \leq \tfrac{1}{2}M_{tt}\Delta t + \tfrac{1}{12}M_x^{(iv)}\Delta x^2$$

The scheme is said to be first-order in t (i.e. $O(\Delta t)$) and second-order in x (i.e. $O(\Delta x^2)$).