Lecture 4

§2.7 Stability: Fourier Analysis

IBVP:

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = u^0(x) \end{cases}$$

Linear, Const Coef

↑

Fourier Series

Exact Solution:

$$\begin{cases} u(x,t) = \sum_{m=1}^{\infty} a_m e^{-(m\pi)^2 t} \sin(m\pi x) \\ a_m = 2\int_0^1 u^0(x) \sin(m\pi x)\, dx \end{cases}$$

at mesh points

$$u(x_j, t_n) = \sum_{m=1}^{\infty} a_m e^{-(m\pi)^2 n\Delta t} \sin(m\pi j\Delta x)$$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

ansatz:

~~Ansatz~~:

$$U_j^n = \sum_{m=1}^{\infty} b_m (\lambda_m)^n \sin(m\pi j\Delta x)$$

ICs $\Rightarrow$ $U_j^0 = \sum_{m=1}^{\infty} b_m \sin(m\pi j\Delta x) = u^0(j\Delta x)$

$$\Longrightarrow \quad b_m = * \qquad m = 1, \cdots, \infty$$

$$\frac{1}{\Delta t} \sum_{m=1}^{\infty} b_m (\lambda_m - 1)(\lambda_m)^n \sin(m\pi j\Delta x)$$

$$= \frac{1}{\Delta x^2} \sum_{m=1}^{\infty} b_m (\lambda_m)^n \big[\sin(m\pi(j+1)\Delta x) - 2\sin(m\pi(j)\Delta x) + \sin(m\pi(j-1)\Delta x) \big]$$

$$\frac{1}{\Delta t}\sum_{m=1}^{\infty} b_m(\lambda_m-1)(\lambda_m)^n \sin(m\pi j\Delta x)$$
$$= \frac{1}{\Delta x^2}\sum_{m=1}^{\infty} b_m(\lambda_m)^n\left(2\cos(m\pi\Delta x)-2\right)\sin(m\pi j\Delta x)$$

Orthogonality:

$$\sum_{j=0}^{J} \sin(m\pi j\Delta x)\sin(m'\pi j\Delta x) = \begin{cases} 0 & \text{if } m\neq m' \\ * & \text{if } m=m' \end{cases}$$

$$\Rightarrow \quad \frac{1}{\Delta t} b_m(\lambda_m-1)(\lambda_m)^n = \frac{1}{\Delta x^2} b_m(\lambda_m)^n\left(2\cos(m\pi\Delta x)-2\right)$$

$$\Rightarrow \quad \boxed{\frac{\lambda_m-1}{\Delta t} = \frac{2}{\Delta x^2}\left(\cos(m\pi\Delta x)-1\right)}$$

m any integer
Δx, Δt

Simple Way

$$U_j^n = \lambda^n e^{im\pi j\Delta x} \qquad (m \text{ any integer})$$

Amplification λ_m or λ

$$|\lambda_m| \leq 1$$

$$\lambda = 1 + \frac{2\Delta t}{\Delta x^2}\left[\cos(m\pi\Delta x)-1\right]$$
$$= 1 - \frac{4\Delta t}{\Delta x^2}\sin^2\left(\frac{m\pi\Delta x}{2}\right)$$

$$|\lambda| \leq 1 \Rightarrow \quad -1 \leq \lambda \leq 1$$
$$-1 \leq 1 - \frac{4\Delta t}{\Delta x^2}\sin^2\left(\frac{m\pi\Delta x}{2}\right) \leq 1$$

$$\Rightarrow \quad \frac{4\Delta t}{\Delta x^2} \sin^2\left(\frac{m\pi\Delta x}{2}\right) \leq 2$$

$$\Rightarrow \quad \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \qquad \text{(CFL Condition)}$$

(Courant- Friedrichs- Lewy)

<u>Conclusion</u> If $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$, the solution of the numerical scheme will remain bounded as $\Delta t \to 0$ and $\Delta x \to 0$, i.e.

$$|u_j^n| \leq \text{Const. for } \begin{array}{l} 0 \leq j \leq J \\ 0 \leq n \leq N \end{array}$$

as $\Delta t \to 0$, $\Delta x \to 0$ and $N \cdot \Delta t \leq t_f$.