5. Lecture 5

§2.6 Convergence of the explicit Scheme

Convergence: $u(x,t)$ exact solution of PDE
U_j^n exact solution of Difference Eqn.

$$e_j^n := U_j^n - u(x_j,t_n) \to 0 \ ?$$

as $\Delta t \to 0$, $\Delta x \to 0$ but $j\Delta x$ and $n\Delta t$ kept fixed

$$u_t = u_{xx}$$

$$\frac{U_j^{n+1}-U_j^n}{\Delta t} = \frac{1}{\Delta x^2}\left[U_{j+1}^n - 2U_j^n + U_{j-1}^n\right]$$

$$\frac{u(x_j,t_{n+1})-u(x_j,t_n)}{\Delta t} = \frac{1}{\Delta x^2}\left[u(x_{j+1},t_n) - 2u(x_j,t_n) + u(x_{j-1},t_n)\right] + T(x_j,t_n)$$

$$\frac{e_j^{n+1}-e_j^n}{\Delta t} = \frac{1}{\Delta x^2}\left[e_{j+1}^n - 2e_j^n + e_{j-1}^n\right] - T(x_j,t_n)$$

$$e_j^{n+1} = e_j^n + \nu\left(e_{j+1}^n - 2e_j^n + e_{j-1}^n\right) - T(x_j,t_n)\Delta t$$

$$e_j^{n+1} = \nu e_{j+1}^n + (1-2\nu)e_j^n + \nu e_{j-1}^n - T(x_j,t_n)\Delta t$$

Define: $E^n := \max_{0\le j\le J} |e_j^n|$

$$|e_j^{n+1}| \le \nu|e_{j+1}^n| + |1-2\nu|\cdot|e_j^n| + \nu|e_{j-1}^n| + |T(x_j,t_n)|\Delta t$$

$$\cancel{|e_j^{n+1}| \le (2\nu + |1-2\nu|)E^n + \max_{0\le j\le J}|T(x_j,t_n)|\Delta t}$$

Define $T := \max_{\substack{0 \le j \le J \\ 0 \le n \le N}} |T(x_j, t_n)|$

$$|e_j^{n+1}| \le \nu E^n + |1-2\nu| E^n + \nu E^n + T \cdot \Delta t$$
$$= (2\nu + |1-2\nu|) E^n + T \cdot \Delta t$$

$$E^{n+1} \le (2\nu + |1-2\nu|) E^n + T \cdot \Delta t$$

$$E^n \le (2\nu + |1-2\nu|) E^{n-1} + T \cdot \Delta t$$

$$\Rightarrow \quad E^{n+1} \le (2\nu + |1-2\nu|)^2 E^{n-1} + (2\nu + |1-2\nu|) T \cdot \Delta t + T \cdot \Delta t$$
$$\le (2\nu + |1-2\nu|)^{n+1} E^0 + (2\nu + |1-2\nu|)^n T \cdot \Delta t + \cdots + (2\nu + |1-2\nu|) \Delta t + T \cdot \Delta t$$

$$2\nu + |1-2\nu| \le 1 \Rightarrow \boxed{\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}}$$

$$E^{n+1} \le E^0 + (n+1)\Delta t \, T \le E^0 + t_f T$$

$$\boxed{E^{n+1} \le E^0 + t_f T}$$

$$E^0 = 0$$

$$T(x,t) = \frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} - \frac{1}{\Delta x^2}\left[u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)\right]$$

$$= \frac{1}{2}u_{tt}(x,\eta)\Delta t - \frac{1}{12}u_x^{(iv)}(\xi,t)\Delta x^2$$

where $t<\eta<t+\Delta t$, $x<\xi<x+\Delta x$

$$|T(x,t)| \le \frac{1}{2}|u_{tt}(x,\eta)|\Delta t + \frac{1}{12}|u_x^{(iv)}(\xi,t)|\Delta x^2$$

$$\le \frac{1}{2}M_{tt}\Delta t + \frac{1}{12}M_x^{(iv)}\Delta x^2$$

$$M_{tt} = \max_{\substack{0\le x\le 1\\ 0\le t\le t_f}}|u_{tt}(x,t)|, \quad M_x^{(iv)} = \max_{\substack{0\le x\le 1\\ 0\le t\le t_f}}|u_x^{(iv)}(x,t)|$$

$$T \le \frac{1}{2}M_{tt}\Delta t + \frac{1}{12}M_x^{(iv)}\Delta x^2$$

$$\boxed{E^{n+1} \le \frac{1}{2}M_{tt}\Delta t + \frac{t_f}{12}M_x^{(iv)}\Delta x^2 \quad \text{if } \frac{\Delta t}{\Delta x^2} \le \frac{1}{2}}$$

$$E^n = O(\Delta t) + O(\Delta x^2) \quad \text{as } \Delta t \to 0,\ \Delta x \to 0 \text{ and } \frac{\Delta t}{\Delta x^2} \le \frac{1}{2}.$$

Conclusion:

(i) $T(x,t) = O(\Delta t) + O(\Delta x^2)$

(ii) Stable if $\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$ (CFL)

(iii) $\|Error\|_\infty = O(\Delta t) + O(\Delta x^2)$ if $\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$.