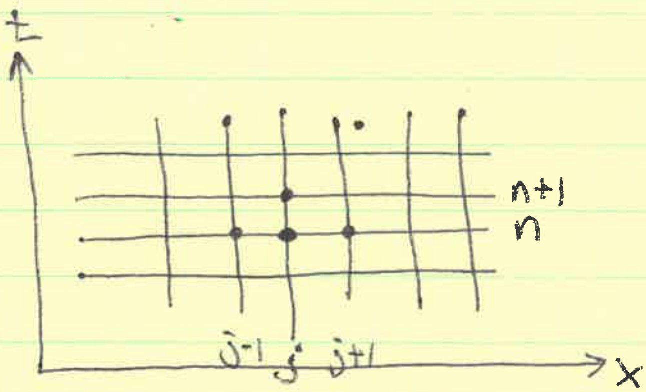
Lecture 6

§2.8 and §2.10: Implicit and θ- methods



Recall Explicit method

based on (x_j, t_n)

$$\frac{\partial u}{\partial t}(x_j,t_n) = \frac{\partial^2 u}{\partial x^2}(x_j,t_n)$$

forward: $\frac{\Delta_{+t} U_j^n}{\Delta t} \sim \frac{\partial u}{\partial t}(x_j,t_n)$

Central but on level $t=t_n$:

$$\frac{\delta_x^2 U_j^n}{\Delta x^2} \sim \frac{\partial^2 u}{\partial x^2}(x_j,t_n)$$

$$\Rightarrow \quad \frac{\Delta_{+t} U_j^n}{\Delta t} = \frac{\delta_x^2 U_j^n}{\Delta x^2}$$

An Implicit Method

based on (x_j, t_{n+1}): $\frac{\partial u}{\partial t}(x_j,t_{n+1}) = \frac{\partial^2 u}{\partial x^2}(x_j,t_{n+1})$

Backward $\frac{\Delta_{-t} U_j^{n+1}}{\Delta t} \sim \frac{\partial u}{\partial t}(x_j,t_{n+1})$

Central but on level $t=t_{n+1}$

$$\frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} \sim \frac{\partial^2 u}{\partial x^2}(x_j,t_{n+1})$$

$$\Rightarrow \quad \frac{\Delta_{-t} U_j^{n+1}}{\Delta t} = \frac{\delta_x^2 U_j^{n+1}}{\Delta x^2}$$

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\Delta t} = \dfrac{1}{\Delta x^2}\left[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}\right] & j = 1, \cdots, J-1 \\ U_0^{n+1} = 0, \quad U_J^{n+1} = 0 \end{cases}$$

How to solve this system for $\{U_j^{n+1}\}_{j=1}^{J-1}$?
Why implicit? Advantage?

The θ-method.

It is often useful to ~~consider~~ construct the difference equation based on at a half point.

Based on $(x_j, t_{n+1/2})$.

$$\frac{\partial u}{\partial t}(x_j, t_{n+1/2}) = \frac{\partial^2 u}{\partial x^2}(x_j, t_{n+1/2})$$

Time derivative — central

$$\frac{\partial u}{\partial t}(x_j, t_{n+1/2}) \sim \frac{\delta_t U_j^{n+1/2}}{\Delta t} = \frac{U_j^{n+1} - U_j^n}{\Delta t}$$

Spatial derivative

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_{n+1/2}) \sim ?$$

known: $\dfrac{\delta_x^2 U_j^{n+1}}{\Delta x^2}$, $\dfrac{\delta_x^2 U_j^n}{\Delta x^2}$

average:

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_{n+1/2}) \sim \theta \frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} + (1-\theta) \frac{\delta_x^2 U_j^n}{\Delta x^2}$$

$$0 \leq \theta \leq 1$$

$$\Rightarrow \quad \frac{\delta_t U_j^{n+1/2}}{\Delta t} = \theta \frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} + (1-\theta) \frac{\delta_x^2 U_j^n}{\Delta x^2}$$

$\theta = 0$: Explicit (Forward)

$\theta = 1$: Backward

$\theta = \frac{1}{2}$, Crank-Nicolson

HW #4

Truncation Error

Exercise 1 (due)

Prove that for sufficiently smooth function $u(x)$ the following statement is true.

$$\frac{u(x+\Delta x)+u(x)}{2} = u(x+\tfrac{1}{2}\Delta t) + O(\Delta x^2)$$

Exercise 2 (due)

Prove that the truncation error of the θ-method is

$$T(x,t) = \begin{cases} O(\Delta t, \Delta x^2) & \text{if } \theta \neq \frac{1}{2} \\ O(\Delta t^2, \Delta x^2) & \text{if } \theta = \frac{1}{2}. \end{cases}$$

Stability

$$U_j^n = \lambda^n e^{i\pi k j \Delta x}$$

$$\frac{\delta_t U_j^{n+1/2}}{\Delta t} = \frac{1}{\Delta t}\left[U_j^{n+1} - U_j^n\right]$$

$$= \frac{\lambda - 1}{\Delta t}\lambda^n e^{i\pi j \Delta x k}$$

$$\frac{\delta_x^2 U_j^n}{\Delta x^2} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

$$= \frac{e^{i\pi\Delta x k} - 2 + e^{-i\pi\Delta x k}}{\Delta x^2}\lambda^n e^{i\pi j \Delta x k}$$

$$= \frac{2\cos(k\pi\Delta x) - 2}{\Delta x^2}\lambda^n e^{i\pi j \Delta x k}$$

$$= -\frac{4\sin^2\left(\frac{k\pi\Delta x}{2}\right)}{\Delta x^2}\lambda^n e^{i\pi j\Delta x k}$$

$$\frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} = -\frac{4\sin^2\left(\frac{k\pi\Delta x}{2}\right)}{\Delta x^2}\lambda^{n+1} e^{i\pi j\Delta x k}$$

$$\frac{\lambda-1}{\Delta t} = -\theta\frac{4\sin^2\left(\frac{k\pi\Delta x}{2}\right)}{\Delta x^2}\lambda-(1-\theta)\frac{4\sin^2\left(\frac{k\pi\Delta x}{2}\right)}{\Delta x^2}$$

$$\left[1+\theta\nu 4\sin^2\left(\frac{k\pi\Delta x}{2}\right)\right]\lambda = 1-(1-\theta)4\nu\sin^2\left(\frac{k\pi\Delta x}{2}\right)$$

$$\lambda = \frac{1-4(1-\theta)\nu\sin^2\left(\frac{k\pi\Delta x}{2}\right)}{1+4\theta\nu\sin^2\left(\frac{k\pi\Delta x}{2}\right)}$$

$$|\lambda|\leq 1 \Rightarrow 4\nu(1-2\theta)\sin^2\left(\frac{k\pi\Delta x}{2}\right)\leq 2$$

$$\Rightarrow \nu(1-2\theta)\leq\frac{1}{2}$$

$$\begin{cases}\theta=0 \to \nu\leq\frac{1}{2}. & \frac{\Delta t}{\Delta x^2}\leq\frac{1}{2}\\ \theta\geq\frac{1}{2} & \text{no restriction}\end{cases}$$

Stability analysis ~~and~~ in L^∞ norm

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