Lecture 7

§2.11 Convergence of the θ-method

$$\nu = \frac{\Delta t}{\Delta x^2} \qquad \nu(1-2\theta) \le \frac{1}{2} \quad \text{(stability)}$$

$$\frac{\delta_t U_j^{n+1/2}}{\Delta t} = \theta \frac{\delta_x^2 U_j^{n+1}}{\Delta x^2} + (1-\theta)\frac{\delta_x^2 U_j^n}{\Delta x^2}$$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{\theta}{\Delta x^2}\left[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}\right] + \frac{1-\theta}{\Delta x^2}\left[U_{j+1}^n - 2U_j^n + U_{j-1}^n\right]$$

$$\frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\Delta t} = \frac{\theta}{\Delta x^2}\left[u(x_{j+1},t_{n+1}) - 2u(x_j,t_{n+1}) + u(x_{j-1},t_{n+1})\right] + \frac{1-\theta}{\Delta x^2}\left[u(x_{j+1},t_n) - 2u(x_j,t_n) + u(x_{j-1},t_n)\right] + T(x_j,t_n)$$

$$\frac{e_j^{n+1} - e_j^n}{\Delta t} = \frac{\theta}{\Delta x^2}\left[e_{j+1}^{n+1} - 2e_j^{n+1} + e_{j-1}^{n+1}\right] + \frac{1-\theta}{\Delta x^2}\left[e_{j+1}^n - 2e_j^n + e_{j-1}^n\right] - T(x_j,t_n)$$

$$-\theta\nu e_{j+1}^{n+1} + (1+2\theta\nu)e_j^{n+1} - \theta\nu e_{j-1}^{n+1} = (1-\theta)\nu e_{j+1}^n + (1-2(1-\theta)\nu)e_j^n + (1-\theta)\nu e_{j-1}^n - T(x_j,t_n)\Delta t$$

$$(1+2\theta\nu)e_j^{n+1} = \theta\nu\left[e_{j+1}^{n+1} + e_{j-1}^{n+1}\right] + (1-\theta)\nu\, e_{j+1}^n + \left[1-2(1-\theta)\nu\right]e_j^n + (1-\theta)\nu\, e_{j-1}^n - T(x_j,t_n)\Delta t$$

$$T := \max_{\substack{0\le x\le 1\\ 0\le t\le t_f}} |T(x,t)|$$

$$E^n := \max_{0\le j\le J} |e_j^n|$$

$$(1+2\theta\nu)|e_j^{n+1}| \le \theta\nu\cdot 2\cdot E^{n+1} + |(1-\theta)\nu|\cdot E^n + |\left[1-2(1-\theta)\nu\right]|\cdot E^n + |(1-\theta)\nu|\, E^n + T\Delta t$$

$$\le 2\theta\nu E^{n+1} + E^n + T\Delta t \qquad \boxed{1-2(1-\theta)\nu \ge 0}$$

$$(1+2\theta\nu)E^{n+1} \le 2\theta\nu E^{n+1} + E^n + T\Delta t$$

$$E^{n+1} \le E^n + T\Delta t$$

$$\le E^0 + t_f\cdot T = t_f T$$

$$T = O(\Delta t, \Delta x^2) \text{ if } \theta \ne \tfrac{1}{2}$$

$$= O(\Delta t^2, \Delta x^2) \text{ if } \theta = \tfrac{1}{2}$$

$$\therefore\ E^{n+1} \le O(\Delta t, \Delta x^2) \text{ if } \theta \ne \tfrac{1}{2}$$

$$\le O(\Delta t^2, \Delta x^2) \text{ if } \theta = \tfrac{1}{2}.$$

$$\boxed{(1-\theta)\nu \le \frac{1}{2}} \qquad (1)$$

Stability condition

$$(1-2\theta)\nu \le \frac{1}{2} \qquad (2)$$

There is a big gap between (1) and (2).

See Fig 2.8.

§2.9 The Thomas algorithm

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{\theta}{\Delta x^2}\left[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}\right] + \frac{(1-\theta)}{\Delta x^2}\left[U_{j+1}^{n} - 2U_j^{n} + U_{j-1}^{n}\right] \qquad j = 1, \cdots, J-1$$

$$U_0^{n+1} = 0, \quad U_J^{n+1} = 0$$

$$-\theta\nu U_{j-1}^{n+1} + (1 + 2\theta\nu)U_j^{n+1} - \theta\nu U_{j+1}^{n+1} = (1-\theta)\nu U_{j-1}^{n} + (1-2(1-\theta)\nu)U_j^n + (1-\theta)\nu U_{j+1}^n$$

$$j = 1, 2, \cdots, J-1$$

$$U_0^{n+1} = 0, \quad U_J^{n+1} = 0$$

Define

$$a_j = -\theta\nu$$
$$b_j = 1+2\theta\nu$$
$$c_j = -\theta\nu$$
$$d_j = (1-\theta)\nu U_{j-1}^n + [1-2(1-\theta)\nu]U_j^n + (1-\theta)\nu U_{j+1}^n$$
$$j = 1, \cdots, J-1$$

$$v_j = U_j^{n+1}$$

$$a_j v_{j-1} + b_j v_j + c_j v_{j+1} = d_j$$
$$j = 1, 2, \cdots, J-1$$

Define

$$a_0 = 0,\ b_0 = 1,\ c_0 = 0,\ d_0 = 0$$
$$a_J = 0,\ b_J = 1,\ c_J = 0,\ d_J = 0$$

BCs $\Rightarrow$
$$a_0 v_{-1} + b_0 v_1 + c_0 v_2 = d_0$$
$$a_J v_{J-1} + b_J v_J + c_J v_{J+1} = d_J$$

$$\boxed{\begin{array}{r} a_j v_{j-1} + b_j v_j + c_j v_{j+1} = d_j \\ j = 0, 1, \cdots, J \end{array}}$$

Let

$$a_j = -\theta\nu$$

$$b_j = 1 + 2\theta\nu$$

$$c_j = -\theta\nu$$

$$d_j = (1-\theta)\nu U_{j-1}^n + (1 - 2(1-\theta)\nu) U_j^n + (1-\theta)\nu U_{j+1}^n$$

$$j = 1, \cdots, J-1$$

$$a_0 = 0, \; b_0 = 1, \; c_0 = 0, \; d_0 = 0$$

$$a_J = 0, \; b_J = 1, \; c_J = 0, \; d_J = 0$$

$$\boxed{a_j U_{j-1} + b_j U_j + c_j U_{j+1} = d_j \qquad j = 0, \cdots, J}$$

$$\boxed{b_j \geq |a_j| + |c_j| \quad \text{for } j = 0, \cdots, J}$$

$$a = \begin{bmatrix} a_0 \\ \vdots \\ \vdots \\ a_J \end{bmatrix}, \quad b = \begin{bmatrix} b_0 \\ \vdots \\ \vdots \\ b_J \end{bmatrix}, \quad c = \cdots \quad \mathbf{U} = \begin{bmatrix} U_0 \\ \vdots \\ U_J \end{bmatrix}$$

$$\begin{bmatrix} b_0 & c_0 & & & \\ a_1 & b_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{J-1} & b_{J-1} & c_{J-1} \\ & & & a_J & b_J \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{J-1} \\ U_J \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{J-1} \\ d_J \end{bmatrix}$$

Matrix Representation

$$a = \begin{bmatrix} a_0 \\ \vdots \\ a_J \end{bmatrix}, \; b, c, d, \; v = \begin{bmatrix} v_0 \\ \vdots \\ v_J \end{bmatrix}$$

$$A = \begin{bmatrix} b_0 & c_0 & & & \\ a_1 & b_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{J-1} & b_{J-1} & c_{J-1} \\ & & & a_J & b_J \end{bmatrix}$$

diagonally dominant

$|b_j| \geq |a_j| + |c_j|$

A: nonsingular

$$\boxed{Av = d}$$

Solution method based on LU decomposition: I

$$A = LU$$

$$L = \begin{bmatrix} l_{00} & & & \\ l_{10} & l_{11} & & \\ & \ddots & \ddots & \\ & & l_{J,J-1} & l_{J,J} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & u_{01} & & \\ & 1 & u_{12} & \\ & & \ddots & \ddots \\ & & & u_{J-1,J} \\ & & & 1 \end{bmatrix}$$

① $l_{00} = b_0$

$l_{00} u_{01} = c_0 \Rightarrow u_{01} = \dfrac{c_0}{l_{00}}$

$l_{10} = a_1$

2. $l_{10}u_{01} + l_{11} = b_1 \Rightarrow l_{11} = b_1 - l_{10}u_{01}$

$l_{11}u_{12} = c_1 \Rightarrow u_{12} = \dfrac{c_1}{l_{11}}$

$l_{21} = a_2$

$\vdots$

$$\begin{cases} l_{ii} = b_i - l_{i,i-1}u_{i-1,i} \\ u_{i,i+1} = \dfrac{c_i}{l_{ii}} \\ l_{i+1,i} = a_{i+1} \end{cases} \qquad i = 0, 1, \cdots, J$$

<u>Remark:</u> $i=0$ and $i=J$

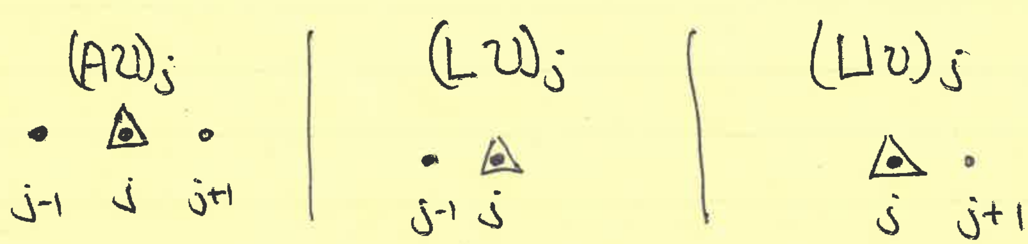
$$LUv = d \Rightarrow Ly = d \Rightarrow Uv = y$$

Solution method based on LU decomposition: II: Using information of the FD Stencil

$$(Av)_j = a_j v_{j-1} + b_j v_j + c_j v_{j+1}$$

$$(Lv)_j = l_{j,j-1} v_{j-1} + l_{j,j} v_j$$

$$(Uv)_j = v_j + u_{j,j+1} v_{j+1}$$



$$LU = A$$

$$\Rightarrow \quad (LUv)_j = (Av)_j$$

$$
\begin{aligned}
(LUv)_j &= [L(Uv)]_j \\
&= l_{j,j-1}(Uv)_{j-1} + l_{j,j}(Uv)_j \\
&= l_{j,j-1}[v_{j-1} + u_{j-1,j} v_j] \\
&\quad + l_{j,j}[v_j + u_{j,j+1} v_{j+1}] \\
&= l_{j,j-1} v_{j-1} + (l_{j,j-1} u_{j-1,j} + l_{j,j}) v_j \\
&\quad + l_{j,j} u_{j,j+1} v_{j+1}
\end{aligned}
$$

$$\Rightarrow \quad \left\{ \begin{aligned} l_{j,j-1} &= a_j \\ l_{j,j} &= b_j - l_{j,j-1} u_{j-1,j} \\ u_{j,j+1} &= \frac{c_j}{l_{j,j}} \end{aligned} \right. \qquad (1)$$

$$j = 0, 1, \cdots, J.$$

- $Ly = d$: $\Rightarrow (Ly)_j = d_j$ (Forward)

$$y_j = \frac{1}{l_{j,j}} \left[d_j - l_{j,j-1} y_{j-1} \right] \qquad (2)$$

$$j = 0, 1, \cdots, J$$

- $LU v = y \Rightarrow (LUv)_j = y_j$

$$v_j = y_j - u_{j,j+1} v_{j+1} \quad \text{(Backward)} \quad (3)$$

$$j = J, J-1, \cdots, 0$$

$\boxed{\text{Summary}}$

Step 1 Compute a_j, b_j, c_j, d_j

Step 2 LU Decomposition

$$l_{j,j-1} = a_j$$
$$l_{j,j} = b_j - l_{j,j-1} u_{j-1,j}$$
$$u_{j,j+1} = \frac{c_j}{l_{j,j}} \qquad j = 0, 1, \cdots, J$$

Remark 1 when $j=0$, do not take account of $l_{j,j-1} u_{j-1,j}$

Remark 2 when $j = J$, do not compute $u_{j,j+1}$

Step 3 Forward (Compute $Ly = d$)

$$y_j = \frac{1}{l_{j,j}} \left[d_j - l_{j,j-1} y_{j-1} \right] \qquad j = 0, 1, \cdots, J$$

Remark 3 when $j = 0$, forget $l_{j,j-1} v_{j-1}$ term.

Step4 Backward (compute $Lu = y$)

$$v_j = y_j - u_{j,j+1} v_{j+1} \qquad j = J, J-1, \cdots, 0$$

Remark 4 when $j = J$. forget term $u_{j,j+1} v_{j+1}$

$$Av = d \qquad A \in \mathbb{R}^{N \times N}$$

$$A = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & & \ddots & & \\ & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & a_N & b_N \end{bmatrix}$$

$$a_j v_{j-1} + b_j v_j + c_j v_{j+1} = d_j$$

$$a_1 = 0, \; c_N = 0 \qquad j = 1, \cdots, N$$

$\boxed{\text{The Thomas algorithm}}$

~~Note define four arrays a, b, c, d~~

INPUT $a, b, c,$ and d; N

Step1 (LU decomposition)

Repeat $j = 1, \cdots, N$ do

$a_j := a_j$;

If $j = 1$ then $b_j := b_j$;

else $b_j := b_j - a_j c_{j-1}$; end

$c_j := \dfrac{c_j}{b_j}$;

end

Step 2 (Forward Step)

Repeat $j = 1, \cdots, N$ **do**

If $j=1$ **then** $d_j := \frac{d_j}{b_j}$

else $d_j := \frac{1}{b_j}(d_j - a_j d_{j-1})$

end

end

Step 3 (Backward Step)

Repeat $j = N, N-1, \cdots, 1$ **do**

If $j = N$ **then** $d_j := d_j$

else $d_j := d_j - c_j d_{j+1}$

end

end

Output solution d

Note 1 Four arrays: a, b, c, and d.

Note 2 On output, the input data of ~~input~~ a, b, c, and d are destroyed; they store the LU decomposition data:

$$a_j := \ell_{j,j-1}$$
$$b_j := \ell_{j,j}$$
$$c_j := u_{j,j+1}.$$