Hour 10 Lecture 9

Conservations and Symmetries
§2.14 and §2.15

Symmetries:

A general parabolic equation may often appear in the self-adjoint form (divergence) form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(p(x,t)\frac{\partial u}{\partial x}\right) \qquad p(x,t) > 0$$

$$u(0,t) = 0, \quad u(1,t) = 0$$

Self-adjoint operator:

$$L = \frac{\partial}{\partial x} p(x,t) \frac{\partial}{\partial x}$$

$$Lu = \frac{\partial}{\partial x} p(x,t)\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(p(x,t)\frac{\partial u}{\partial x}\right)$$

define $\quad (u,v) = \int_0^1 u(x,t)\, v(x,t)\, dx$

for u, v: $u(0,t) = 0 = u(1,t) = v(0,t) = v(1,t)$

if $(Lu, v) = (u, Lv)$ for any u and v

then L is called a self-adjoint operator.

① $$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{1}{\Delta x}\left[p_{j+\frac{1}{2}}^{n+1}\frac{U_{j+1}^{n+1} - U_j^{n+1}}{\Delta x} - p_{j-\frac{1}{2}}^{n+1}\frac{U_j^{n+1} - U_{j-1}^{n+1}}{\Delta x}\right]$$

$$-\nu p_{j-\frac{1}{2}}^{n+1} U_{j-1}^{n+1} + \left(1 + \nu p_{j-\frac{1}{2}}^{n+1} + \nu p_{j+\frac{1}{2}}^{n+1}\right) U_j^{n+1} - \nu p_{j+\frac{1}{2}}^{n+1} U_{j+1}^{n+1}$$

$$= U_j^n$$

$A = \text{tridiag}\left(-\nu p_{j-\frac{1}{2}}^{n+1},\ 1 + \nu p_{j-\frac{1}{2}}^{n+1} + \nu p_{j+\frac{1}{2}}^{n+1},\ \nu p_{j+\frac{1}{2}}^{n+1}\right)$

symmetric positive definite matrix.

(2) $$\frac{\partial u}{\partial t} = p(x,t)\frac{\partial^2 u}{\partial x^2} \cancel{+ p_x'(x,t)\frac{\partial u}{\partial x}}$$

non-symmetric coeficient matrix !

<u>Conservations</u>

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$$

$$\frac{d}{dt}\int_0^1 u(x,t)\,dt = \int_0^1 \frac{\partial^2 u}{\partial x^2}\,dx = 0$$

$$H(t) = \int_0^1 u(x,t)\,dt = \sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}} u(x,t)\,dt$$

discrete analogue:

$$H_d(t_n) \approx \sum_{j=0}^{J-1} \Delta x\, U_j^n$$

$$H_d(t_n) = \sum_{j=0}^{J-1} \Delta x \cdot \frac{1}{2}\left(U_j^n + U_{j+1}^n\right)$$

$$H_d(t_0) = H_d(t_1) = \cdots = H_d(t_n) \; ?$$