HW #5

Prob: 2.7 38+1

Homework Lecture 10

Nonlinear Equations §2.17

Burgers' Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = u^0(x) = \sin(\pi x)$$

Explicit Scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + U_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = \frac{\varepsilon}{\Delta x^2}\left[U_{j+1}^n - 2U_j^n + U_{j-1}^n\right]$$

Implicit Scheme (Backward)

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\Delta t} + U_j^{n+1} \dfrac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} = \dfrac{\varepsilon}{\Delta x^2}\left[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}\right] \\ \qquad j = 1, \cdots, J-1 \\ U_0^{n+1} = U_J^{n+1} = 0 \end{cases}$$

Nonlinear PDEs
Part I: Newton Iteration

Motivation: Scalar equation

$$f(x) = 0$$

$x^{(0)}$: approximation

x^e: exact solution

Taylor expansion

$$f(x^e) = 0 \approx f(x^{(0)}) + f'(x^{(0)})(x^e - x^{(0)})$$

$$\Rightarrow \quad x^e \approx x^{(0)} - [f'(x^{(0)})]^{-1} f(x^{(0)})$$

$$x^{(k+1)} = x^{(k)} - [f'(x^{(0)})]^{-1} f(x^{(k)})$$

Vectored equation

$$f(v) := \begin{bmatrix} f_1(v_1, \cdots, v_n) \\ f_2(v_1, \cdots, v_n) \\ \vdots \\ f_n(v_1, \cdots, v_n) \end{bmatrix} = 0$$

$$\boxed{v^{(k+1)} = v^{(k)} - [J(v^{(k)})]^{-1} f(v^{(k)})}$$

or

$$\boxed{\begin{aligned} J(v^{(k)})\,\Delta v^{(k)} &= -f(v^{(k)}) \\ v^{(k+1)} &= v^{(k)} + \Delta v^{(k)} \end{aligned}}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial v_1} & \cdots & \frac{\partial f_1}{\partial v_n} \\ \frac{\partial f_n}{\partial v_1} & \cdots & \frac{\partial f_n}{\partial v_n} \end{bmatrix}$$

Modified Newton Iteration:

$H_k \approx J(v^{(k)})$. easy to compute

$$\boxed{\begin{aligned} H_k \Delta v^{(k)} &= -f(v^{(k)}) \\ v^{(k+1)} &= v^{(k)} + \Delta v^{(k)} \end{aligned}}$$

Damped Modified Newton Iteration

$$H_k \Delta v^{(k)} = -f(v^{(k)})$$

$$v^{(k+1)} = v^{(k)} + \gamma_k \Delta v^{(k)}$$

$$\gamma_k = \frac{f^T(v^{(k)})\, J \Delta v^{(k)}}{\| J \Delta v^{(k)} \|^2}$$

$$J \Delta v^{(k)} \approx \frac{f\left(v^{(k)} + \frac{\varepsilon}{\|\Delta v^{(k)}\|} \Delta v^{(k)}\right) - f(v^{(k)})}{\frac{\varepsilon}{\|\Delta v^{(k)}\|}}$$

Hour 12 Lecture 11

Nonlinear PDEs
Part II: Burgers' Equation

Burgers' Equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} - u \dfrac{\partial u}{\partial x} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = u^0(x) \end{cases}$$

ε: physical parameter

Explicit Scheme (Forward. (x_j, t_n))

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{\varepsilon}{\Delta x^2}\left[U_{j+1}^n - 2U_j^n + U_{j-1}^n \right] - U_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}$$

Crank-Nicolson Scheme.

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{\varepsilon}{2\Delta x^2} \delta_x^2 U_j^{n+1} - \frac{1}{2} U_j^{n+1} \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} + \frac{\varepsilon}{2\Delta x^2} \delta_x^2 U_j^n - \frac{1}{2} U_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}$$

$\nu = \dfrac{\Delta t}{\Delta x^2}$, $\mu = \dfrac{\Delta t}{\Delta x}$

$$-\frac{\varepsilon\nu}{2}u_{j+1}^{n+1}+[1+\varepsilon\nu]u_j^{n+1}-\frac{\varepsilon\nu}{2}u_{j-1}^{n+1}$$
$$+\frac{\mu}{4}u_j^{n+1}[u_{j+1}^{n+1}-u_{j-1}^{n+1}]=d_j^{n+1}$$

$$d_j^{n+1}=\frac{\varepsilon\nu}{2}u_{j+1}^{n}+(1-\varepsilon\nu)u_j^{n}+\frac{\varepsilon\nu}{2}u_{j-1}^{n}$$
$$-\frac{\mu}{4}u_j^{n}[u_{j+1}^{n}-u_{j-1}^{n}]$$

$v_j :=: u_j^{n+1}$ $\qquad j=0,1,\cdots,J$

$$\begin{cases} f_0 := v_0 \\ f_j := -\frac{\varepsilon\nu}{2}v_{j+1}+(1+\varepsilon\nu)v_j-\frac{\varepsilon\nu}{2}v_{j-1} \\ \qquad +\frac{\mu}{4}v_j(v_{j+1}-v_{j-1})-d_j^{n+1} \qquad j=1,\cdots,J-1 \\ f_J := v_J \end{cases}$$

$$\boxed{f(v)=0 \qquad \text{initial guess: } v^{(0)}=u_j^n}$$

Damped modified Newton iteration:

$$H_k\,\Delta v^{(k)}=-f(v^{(k)})$$
$$v^{(k+1)}=v^{(k)}+\delta_k\,\Delta v^{(k)}$$

Note: if Newton iteration fails, Δt should be reduced in order to guarantee the convergence.

Choice of H_k:

$$J(v) = \begin{bmatrix} 1 & & \\ -\frac{\varepsilon \nu}{2} & 1+\varepsilon\nu & -\frac{\varepsilon\nu}{2} \\ & & \end{bmatrix} + \frac{M}{4}\begin{bmatrix} 0 & & \\ -v_1 & v_2 - v_0 & v_1 \\ & & \end{bmatrix}$$

$$H_k = J(v^{(0)}) = J(U^n)$$

$$H_k = \text{linear part of } J(v).$$

Linearly implicit Scheme $\quad f(x^{n+1}) \approx f(x^n) + f'(x^n)(x^{n+1}-x^n)$

$$U_j^{n+1}\left(U_{j+1}^{n+1} - U_{j-1}^{n+1}\right) \approx U_j^n\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$+ \left(U_{j+1}^n - U_{j-1}^n\right)\left(U_j^{n+1} - U_j^n\right)$$

$$+ U_j^n\left[U_{j+1}^{n+1} - U_{j-1}^{n+1} - \left(U_{j+1}^n - U_{j-1}^n\right)\right]$$

$$\frac{U_j^{n+1}-U_j^n}{\Delta t} = \frac{\varepsilon}{2\Delta x^2}\delta_x^2\left(U_j^{n+1}-U_j^n\right) - \frac{1}{4\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)\cdot$$
$$\left(U_j^{n+1}-U_j^n\right) - \frac{1}{4\Delta x}U_j^n\left[\left(U_{j+1}^{n+1}-U_{j-1}^{n+1}\right)-\left(U_{j+1}^n - U_{j-1}^n\right)\right]$$
$$+\frac{\varepsilon}{\Delta x^2}\delta_x^2 U_j^n - U_j^n\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}$$

good stability!

HW6

Burgers' equation

$$\begin{cases} u_t = \varepsilon u_{xx} - \left(\frac{u^2}{2}\right)_x & 0<x<1 \\ u(0,t) = u_{exact}(0,t) \\ u(1,t) = u_{exact}(0,t) \\ u(x,0) = u_{exact}(x,0) \end{cases}$$

Requirement: Linearized CN

① plot $\|e\|_\infty(t=1) \propto J$ show convergence order.

② Solution $\propto x$ for $t = 0.25, 0.5, 0.75$ and $t=1$.

③ Study $\varepsilon = 10^{-3}$?

$$u_{exact} = \frac{0.1 r_1 + 0.5 r_2 + r_3}{r_1 + r_2 + r_3}$$

$$r_1 = e^{\frac{-x+0.5-4.95t}{20\varepsilon}}$$

$$r_2 = e^{\frac{-x+0.5-0.75t}{4\varepsilon}}$$

$$r_3 = e^{\frac{-x+0.375}{2\varepsilon}}$$

$\varepsilon = 0.1$ $\quad t: [0,1]$.

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