Midterm Exam

1. The CN Scheme

$$\frac{\partial u}{\partial t}(x_j, t_{n+1/2}) = \frac{\partial^2 u}{\partial x^2}(x_j, t_{n+1/2}) + f(x_j, t_{n+1/2})$$

$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{1}{2}\,\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$+ \frac{1}{2}\,\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + f_j^{n+1/2}$$

where $f_j^{n+1/2} = f(x_j, t_{n+1/2})$

2. local truncation error

$$\tau_j^n = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t}$$

$$- \frac{1}{2}\,\frac{u(x_{j+1}, t_n) - 2\,u(x_j, t_n) + u(x_{j-1}, t_n)}{\Delta x^2}$$

$$- \frac{1}{2}\,\frac{u(x_{j+1}, t_{n+1}) - 2\,u(x_j, t_{n+1}) + u(x_{j-1}, t_{n+1})}{\Delta x^2}$$

$$- f(x_j, t_{n+1/2})$$

Using the Taylor expansion $\Rightarrow$

$\cdots$

$$\tau_j^n = O(\Delta t^2) + O(\Delta x^2)$$

The CN Scheme is Consistent with the PDE.

3. Stability analysis

Stability is about the boundedness of the difference between two solutions with different initial solutions

u_j^n :

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t} = \dfrac{1}{2}\dfrac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}+\dfrac{1}{2}\dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2} \\ \qquad + f_j^{n+\frac{1}{2}} \\ u_j^0 \text{ initial solution} \end{cases}$$

v_j^n :

$$\begin{cases} v_j^{n+1}-v_j^n = \dfrac{1}{2}\dfrac{v_{j+1}^n-2v_j^n+v_{j-1}^n}{\Delta x^2}+\dfrac{1}{2}\dfrac{v_{j+1}^{n+1}-2v_j^{n+1}+v_{j-1}^{n+1}}{\Delta x^2} \\ \qquad + f_j^{n+\frac{1}{2}} \\ \text{initial solution : } v_j^0 \end{cases}$$

Let $w_j^n = u_j^n - v_j^n$

$\Rightarrow$

$$\begin{cases} \dfrac{w_j^{n+1}-w_j^n}{\Delta t} = \dfrac{1}{2}\dfrac{w_{j+1}^n-2w_j^n+w_{j-1}^n}{\Delta x^2}+\dfrac{1}{2}\dfrac{w_{j+1}^{n+1}-2w_j^{n+1}+w_{j-1}^{n+1}}{\Delta x^2} \\ w_j^0 = u_j^0 - v_j^0 \end{cases}$$

3(A): Fourier method:

Let $w_j^n = \lambda^n e^{i\pi m x_j}$

$$\Rightarrow \frac{\lambda-1}{\Delta t} = \frac{-4}{2\Delta x^2}\sin^2\left(\frac{\pi m \Delta x}{2}\right) - \frac{4\lambda}{2\Delta x^2}\sin^2\left(\frac{\pi m \Delta x}{2}\right)$$

$$\lambda = \frac{1 - \frac{2\Delta t}{\Delta x^2}\sin^2\left(\frac{m\pi\Delta x}{2}\right)}{1 + \frac{2\Delta t}{\Delta x^2}\sin^2\left(\frac{m\pi\Delta x}{2}\right)}$$

Easy to show: $-1 \le \lambda \le 1$

$\Rightarrow$ ~~uncon~~ $\boxed{\text{unconditionally stable}}$ in Fourier Analysis

3(B) L^∞ analysis:

$$W^n = \max_j |w_j^n|, \quad W^{n+1} = \max_j |w_j^{n+1}|$$

$$\left(1 + \frac{\Delta t}{\Delta x^2}\right) w_j^{n+1} = \frac{\Delta t}{2\Delta x^2} w_{j+1}^n + \left(1 - \frac{\Delta t}{\Delta x^2}\right) w_j^n + \frac{\Delta t}{2\Delta x^2} w_{j-1}^n + \frac{\Delta t}{2\Delta x^2} w_{j+1}^{n+1} + \frac{\Delta t}{2\Delta x^2} w_{j-1}^{n+1}$$

$$\Rightarrow \left(1 + \frac{\Delta t}{\Delta x^2}\right) w_j^{n+1} \le \frac{\Delta t}{2\Delta x^2}|w_{j+1}^n| + \left|1 - \frac{\Delta t}{\Delta x^2}\right| \cdot |w_j^n| + \frac{\Delta t}{2\Delta x^2}|w_{j-1}^n| + \frac{\Delta t}{2\Delta x^2}|w_{j+1}^{n+1}| + \frac{\Delta t}{2\Delta x^2}|w_{j-1}^{n+1}|$$

$$\le \frac{\Delta t}{\Delta x^2} W^n + \left|1 - \frac{\Delta t}{\Delta x^2}\right| W^n + \frac{\Delta t}{\Delta x^2} W^{n+1}$$

or

$$\left(1 + \frac{\Delta t}{\Delta x^2}\right) W^{n+1} \le \left(\frac{\Delta t}{\Delta x^2} + \left|1 - \frac{\Delta t}{\Delta x^2}\right|\right) W^n + \frac{\Delta t}{\Delta x^2} W^{n+1}$$

or

$$W^{n+1} \le \left(\frac{\Delta t}{\Delta x^2} + \left|1 - \frac{\Delta t}{\Delta x^2}\right|\right) W^n$$

If $\frac{\Delta t}{\Delta x^2} \le 1$ or $\Delta t \le \Delta x^2$, we have

$$W^{n+1} \le W^n$$

or $$W^{n+1} \le W^n \le \cdots \le W^0$$

Thus, in L^∞ norm, the scheme is stable if $\Delta t \le \Delta x^2$.

Conditionally stable in L^∞.

4. Let $e_j^n = U_j^n - u(x_j, t_n)$

Using the definition of τ_j^n, we have the error equation as:

$$\frac{e_j^{n+1} - e_j^n}{\Delta t} = \frac{1}{2}\, \frac{e_{j+1}^n - 2e_j^n + e_{j-1}^n}{\Delta x^2} + \frac{1}{2}\, \frac{e_{j+1}^{n+1} - 2e_j^{n+1} + e_{j-1}^{n+1}}{\Delta x^2}$$

$$- \tau_j^n$$

As in L^∞ stability analysis, we have

$$E^{n+1} \le \left(\frac{\Delta t}{\Delta x^2} + \left|1 - \frac{\Delta t}{\Delta x^2}\right| \right) E^n + \Delta t \max_j |\tau_j^n|$$

Thus, if $\Delta t \le \Delta x^2 \Rightarrow$

$$E^{n+1} \le E^n + \Delta t \left(M_1 \Delta t^2 + M_2 \Delta x^2 \right)$$

$$\Rightarrow \quad E^n \le n\Delta t \left(M_1 \Delta t^2 + M_2 \Delta x^2 \right)$$

5. The scheme is unconditionally stable in Fourier analysis, but stable and convergent in L^∞ only under condition $\Delta t \le \Delta x^2$. #